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Uniform Exponential Energy Decay of Wave Equations in a Bounded Region with $L_2(0, \infty; L_2(\Gamma))$-Feedback Control in the Dirichlet Boundary Conditions*

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Given an open bounded domain $\Omega \subset R^n$, $n \geqslant 2$, we consider the wave equation: $w_{tt} = \Delta w$ in $(0, \infty) \times \Omega$, with initial conditions $w(0, x) = w_0(x) \in L_2(\Omega)$, $w_1(0, x) = w_1(x) \in H^{-1}(\Omega)$, and nonhomogeneous boundary condition of *Dirichlet* type $w(t, \sigma) \equiv u(t, \sigma)$ on $(0, \infty) \times \Gamma$, Γ being the boundary of Ω. In contrast with the *unitary* group situation obtained when $u \equiv 0$, we seek to express the nonhomogeneous boundary term $u(t, \sigma)$ as a suitable linear feedback operator F of the velocity w_t: (*) $u(t, \sigma) = Fw_t$ such that (i) $Fw_t \in L_2(0, \infty; L_2(\Gamma))$, (ii) the corresponding closed loop system, obtained by using (*) in the boundary condition generates an s.c. semigroup which decays exponentially as $t \to +\infty$ in the *uniform operator topology* of $L_2(\Omega) \times H^{-1}(\Omega) \equiv Z$: (**) $\|[w(t), w_t(t)]\|_Z \leqslant Me^{-\delta t} \|[w(0), w_t(0)]\|_Z$, $t \geqslant 0$, for some $\delta > 0$. Having identified the candidate $(\partial/\partial\nu)(A^{-1}w_t)$ for Fw_t, $-A$ being the Laplacian Δ with zero Dirichlet B.C., we prove two stabilization results with such a choice of F. First, that for any Ω with only some regularity assumption on Γ (say of class C^1), all feedback closed loop solutions go to zero as $t \to \infty$ in the *strong norm* of Z: $\|[w(t), w_t(t)]\|_Z \to 0$. Secondly, that the much stronger sought after result on the exponential decay (**) in the uniform operator norm is indeed achieved, provided that Ω satisfies, in addition, a further assumption ("vector field condition"). This condition is expressed in terms of the existence of a suitable vector field for $\bar{\Omega}$ and is satisfied, in particular, if Ω is strictly convex or else if $\Omega = \Omega_1 \backslash \Omega_2$ with Ω_1, Ω_2 strictly convex, $\bar{\Omega}_1 \subsetneqq \Omega_2$. As a consequence of the exponential decay (**) in the uniform operator norm of Z of the closed loop feedback system, we obtain—via I. Lasiecka and R. Triggiani, *Appl. Math. Optim.* 10, 275–286) or (I. Lasiecka, J. L. Lions, and R. Triggiani, *J. Math. Pures Appl.*, in press) and (D. L. Russell, Differential games and control theory, Marcel Dekker, New York 1974, 291–319.)—a new, sharper, and complete result of *exact boundary controllability* over the entire (natural) state space Z of the open loop system: for any pair (w_0, w_1) of initial data in Z, there is an $L_2(0, T; L_2((\Gamma))$—Dirichlet (open loop) boundary control u, for a universal

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finite time $T>0$ sufficiently large, such that the corresponding solution satisfies $[w(t), w_t(t)] \in C([0, T]; Z)$ and $w(T) = w_t(T) = 0$, with $\|u\|_{L_2(0,T;L_2(\Gamma))} \leqslant C_T \|[w_0, w_1]\|_Z$.

1. Introduction, Preliminaries, and Statement of Main Results

Let Ω be a bounded, open domain in $R^n (n \geqslant 2)$ with boundary Γ. On Ω, we consider the following second-order hyperbolic mixed problem in $w(t, x)$, $x \in \Omega$, with non-homogeneous Dirichlet data $u(t, \sigma)$, $\sigma \in \Gamma$:

$$\frac{\partial^2 w}{\partial t^2} = \Delta w \qquad \text{in } (0, \infty) \times \Omega \quad \text{(a)}$$

$$w(0, x) = w_0(x) \in L_2(\Omega);\ w_t(0, x) = w_1(x) \in H^{-1}(\Omega), \qquad \text{(b)}$$

$$w(t, \sigma) = u(t, \sigma) \qquad \text{in } (0, \infty) \times \Gamma \quad \text{(c)} \tag{1.1}$$

Regularity results for the solution w of (1.1) [with $-\Delta$ replaced by a general second-order uniformly elliptic operator] were recently obtained in [LT.1–2; L.4] for $u \in L_2(0, T; L_2(\Gamma))$, $0 < T < \infty$, and in [LLT.1] for $u \in H^k(\Sigma) = L_2(0, T; H^k(\Gamma)) \cap H^k(0, T; L_2(\Gamma))$, $k = 0, 1,...$ as well as for intermediate cases; see also [S.1] for regularity results on *the half-space* for $k \geqslant 1$.

If $A = -\Delta$ with homogeneous Dirichlet B.C., then A is a positive self-adjoint operator in $L_2(\Omega)$. Moreover, the operator $\mathscr{A}_0 = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}$ with domain $\mathscr{D}(A^{1/2}) \otimes L_2(\Omega)$ generates a strongly continuous *unitary* group, denoted by $e^{\mathscr{A}_0 t}$, on the space $Z \equiv L_2(\Omega) \otimes H^{-1}(\Omega)$, topologized by

$$\left(\begin{vmatrix} z_1 \\ z_2 \end{vmatrix}, \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} \right)_Z = (z_1, y_1)_\Omega + (z_2, y_2)_{H^{-1}(\Omega)} \tag{1.2}$$

Here and throughout the paper, $(\ ,\ )_\Omega$ denotes the $L_2(\Omega)$-inner product with associated norm $\|\ \|_\Omega$. Also

$$\begin{aligned} H^{-1}(\Omega) &= [H_0^1(\Omega)]' = [\mathscr{D}(A^{1/2})]' && \text{(see (1.8) below)} \\ \|y\|_{H^{-1}(\Omega)} &\equiv \|y\|_{[\mathscr{D}(A^{1/2})]'} \equiv \|A^{-1/2} y\|_\Omega && \text{(norm equivalence)} \end{aligned} \tag{1.3}$$

Thus, the *free* solution of (1.1), i.e., the solutions of (1.1) with $u \equiv 0$, are given by $[w(t), w_t(t)] = e^{\mathscr{A}_0 t}[w_0, w_1]$ on Z; in particular, they do *not* decay to zero. With this well-known fact at hand, we can now state the aim of the present paper.

Formulation of Problem to be Investigated

We shall study the question of existence and construction of a boundary feedback operator F based on the velocity w_t:

$$\begin{gathered} w_t \in H^{-1}(\Omega) \to F(w_t) \in L_2(\Gamma) \\ F(w_t) \in L_2(0, \infty; L_2(\Gamma)) \end{gathered} \tag{1.4}$$

such that the boundary feedback input function $u = F(w_t) \in L_2(0, \infty; L_2(\Gamma))$, inserted in (1.1c), produces a *feedback semigroup, which is exponentially stable in the uniform operator topology of Z.*

Due to the Dirichlet nature of the B.C. (1.1c), the above problem was, to the authors' knowledge, open, while the corresponding problem with u acting in the Neumann B.C. was solved only recently [C.1–2; L.1] (see reference to literature below).

Our *motivation* for studying the described problem is twofold:

(i) on the one hand, this problem is one of the central questions in boundary feedback stabilization theory;

(ii) on the other hand, an affirmative solution to the above problem is a necessary *prerequisite* for studying the *regulator problem* for the hyperbolic dynamics (1.1):

$$\text{minimize } J(u, w) = \int_0^\infty \{ \|w(t)\|_\Omega^2 + \|w_t(t)\|_{H^{-1}(\Omega)}^2 + \|u(t)\|_\Gamma^2 \} \, dt$$

over all $u \in L_2(0, \infty; L_2(\Gamma))$, and related algebraic Riccati equation, needed for the feedback synthesis of the optimal solution, see [LT.3]. As to (i), we remark that a positive solution of the proposed problem will substantially complement the feedback stabilization results of [LT.4]: here the feedback operator (nonlocal, of finite rank) $u(t, \sigma) = (w_t(t, \cdot), g_1(t, \cdot))_\Omega \, g_2(\sigma)$, $g_1 \in L_2(\Omega)$, $g_2 \in L_2(\Gamma)$, was chosen and two results were shown: (a) that for certain classes of vectors g_1, g_2, the feedback system is, indeed, *strongly stable*, while (b) *uniform* stability of the feedback system is, however, ruled out, for all choices of g_1, g_2, due to the finite rank character of the feedback. In the present paper, in contrast, we seek exponential *uniform* stability, via a suitable feedback operator F. To elaborate on (ii), we recall that the study of the regulator problem requires at the very outset the following property ["Finite cost condition"]: that for each pair of initial data $[w_0, w_1] \in Z$, there exists some (open loop) $u \in L_2(0, \infty; L_2(\Gamma))$ such that the corresponding solution of (1.1) satisfies $[w(t), w_t(t)] \in L_2(0, \infty; Z)$

so that the corresponding cost is finite: $J < \infty$. That this finite cost condition is indeed possible in the case of dynamics (1.1) is far from clear. No example was, to our knowledge, known. On the other hand, a positive solution to the proposed problem studied in this paper plainly implies that the finite cost condition holds true for (1.1), thus clearing the way to the study of the corresponding regulator problem. This study is successfully carried out in [LT.3]. *As a result* of this theory, one obtains—remarkably—*another* feedback operator $u^0(t) = -B^*P\,|^{w^0(t)}_{w^0_t(t)}| \in L_2(0, \infty; L_2(\Gamma))$ which likewise produces a s.c. semigroup which decays exponentially in the *uniform operator topology* of Z: here B^* is an operator (of trace type) depending on the dynamics (1.1), P is a Riccati operator, and $u^0(t)$, $[w^0(t), w^0_t(t)]$ the (unique) optimal minimizing pair for J.

Choice of Operator F and Statement of Main Results

We begin with some preliminary background material, needed to both motivate our choice of the operator F and state our main results. We first introduce the Dirichlet map D (natural harmonic extension of boundary data on Γ into the interior Ω) defined by

$$Dv = h, \qquad \text{where } \begin{cases} \Delta h = 0 \text{ in } \Omega \\ h|_\Gamma = v \text{ in } \Gamma. \end{cases} \tag{1.5}$$

It is a well-known result of elliptic theory [LM.1; N.1] that

$$D: \qquad \text{continuous operator } H^s(\Gamma) \to H^{s+1/2}(\Omega),\ s \text{ real} \tag{1.6}$$

We shall in particular use

$$\begin{aligned} &D: \quad \text{continuous } L_2(\Gamma) \to H^{1/2-2\varepsilon}(\Omega) \equiv \mathscr{D}(A^{1/4-\varepsilon}),\ \varepsilon > 0 && \text{(a)} \\ &\|y\|_{H^{1/2-2\varepsilon}(\Omega)} = \|A^{1/4-\varepsilon}y\|_{L_2(\Omega)} && \text{(b)} \end{aligned} \tag{1.7}$$

where for the identification on the right of (1.7a), as well as for the identification

$$H_0^{3/2-2\varepsilon}(\Omega) \equiv \mathscr{D}(A^{3/4-\varepsilon}), \qquad \varepsilon > 0,\ \varepsilon \neq \tfrac{1}{2}, \tag{1.8}$$

we refer to [F1; L2 Appendix] (the identification being set theoretical and topological, in the sense of norm equivalence). If D^* is the adjoint operator of D: $(Dv, y)_{L_2(\Omega)} = (v, D^*y)_{L_2(\Gamma)}$, we have in particular

$$D^*: \qquad \text{continuous } H^{-s}(\Omega) \to H^{-s+1/2}(\Gamma),\ 0 \leqslant s \leqslant \tfrac{1}{2} \tag{1.9}$$

This stems from (1.6) with $0 \leqslant s \leqslant \frac{1}{2}$, where $H^{-1}(\Omega) = [H_0^1(\Omega)]' = [\mathscr{D}(A^{1/2})]'$ via (1.8). A crucial property to be used in the sequel is

$$-D^*Ay = \frac{\partial y}{\partial \nu}, \qquad y \in \mathscr{D}(A) \tag{1.10}$$

which is proved by Green's second theorem (e.g. [LLT1]), where $\partial/\partial\nu$ is the normal derivative (pointed outward). We next recall from [LT1–LT2] that problem (1.1) can be modeled as an abstract second under equation, either in *factor form*

$$w_{tt} = -A[w - Du] \qquad \text{on } L_2(\Omega) \tag{1.11a}$$

or else in *perturbation form*

$$w_{tt} = -Aw + ADu \qquad \text{on } [\mathscr{D}(A)]' \text{ (or even on } [\mathscr{D}(A^{3/4+\varepsilon})]', \text{ by (1.7a))} \tag{1.11b}$$

where A is extended, with the same symbol, as an operator $L_2(\Omega) \to [\mathscr{D}(A)]'$. The first-order version of (1.11b) is

$$\frac{d}{dt}\begin{vmatrix} w \\ w_t \end{vmatrix} = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \begin{vmatrix} w \\ w_t \end{vmatrix} + \begin{vmatrix} 0 \\ ADu \end{vmatrix}, \qquad \text{say on } [\mathscr{D}(A^{1/2})]' \times [\mathscr{D}(A)]'. \tag{1.11c}$$

Since $\left|\begin{smallmatrix} 0 & I \\ -A & 0 \end{smallmatrix}\right|$ is skew-adjoint, Eq. (1.11c) plainly suggests to take as a natural candidate $u = -D^*w_t$, since this choice then makes the corresponding feedback operator

$$\mathscr{A} = \begin{vmatrix} 0 & I \\ -A & -ADD^* \end{vmatrix}, \qquad \mathscr{D}(\mathscr{A}) = \{ y \in Z : \mathscr{A}y \in Z \} \tag{1.12}$$

dissipative on Z: indeed, by (1.12) and (1.3)

$$\begin{aligned} \operatorname{Re}(\mathscr{A}y, y)_Z &= \operatorname{Re}\left(\begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} y, y \right)_Z - (ADD^*y_2, y_2)_{[\mathscr{D}(A^{1/2})]'} \\ &= 0 - \|D^*y_2\|_\Gamma^2, \qquad y \in \mathscr{D}(\mathscr{A}) \end{aligned} \tag{1.13}$$

where here and throughout the paper, $\| \; \|_\Gamma$ is the $L_2(\Gamma)$-norm, $\|x\|_\Gamma^2 = (x, x)_\Gamma$. A more explicit description of $y = [y_1, y_2] \in \mathscr{D}(\mathscr{A})$ is: (i) $y_2 \in L_2(\Omega)$, and (ii) $-A[y_1 + DD^*y_2] \in [\mathscr{D}(A^{1/2})]'$, the latter equivalent to $y_1 + DD^*y_2 \in \mathscr{D}(A^{1/2}) = H_0^1(\Omega)$. Thus

$$\mathscr{D}(\mathscr{A}) = \left\{ y = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} : y_2 \in L_2(\Omega) \text{ and } y_1 + DD^*y_2 \in \mathscr{D}(A^{1/2}) = H_0^1(\Omega) \right\} \tag{1.14}$$

and thus, a fortiori, $y_1 \in H^1(\Omega)$, since

$$DD^* \colon \text{continuous } L_2(\Omega) \to H^1(\Omega) \tag{1.15}$$

by (1.9) for $s=0$, followed by (1.6) for $s=\frac{1}{2}$. With the above choice $u=-D^*w_t$, the open loop system (1.11a) becomes the closed loop system

$$w_{tt}=-A[w+DD^*w_t] \qquad \text{on } L_2(\Omega). \tag{1.16}$$

Note that, by (1.10), the chosen feedback candidate can also be written as

$$u=-D^*w_t=-D^*AA^{-1}w_t=\frac{\partial}{\partial\nu}(A^{-1}w_t). \tag{1.17}$$

The resulting feedback system is

$$\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}=\Delta w && \text{on } (0,\infty)\times\Omega \quad \text{(a)}\\
&w(0,x)=w_0(x)\in L_2(\Omega);\ w_t(0,x)=w_1(x)\in H^{-1}(\Omega) && \text{(b)}\\
&w=-D^*w_t && \text{on } (0,\infty)\times\Gamma \quad \text{(c)}
\end{aligned} \tag{1.18}$$

Our main results concern the decay of the solutions of the feedback system (1.18), either in the *strong* norm of Z or else—with additional assumptions on Ω—on the *uniform* operator norm of $\mathscr{L}(Z)$. The same feedback (1.17) is simply obtained by differentiating the "energy" $E(t)$ in (3.1), using Green's second theorem and imposing $dE/dt\leqslant 0$.

THEOREM 1.1 (Well posedness and strong stabilization). (i) *The operator $\mathscr{A}$ in* (1.12), (1.14) *is dissipative on $Z=L_2(\Omega)\otimes H^{-1}(\Omega)$ and satisfies here: range $(\lambda I-\mathscr{A})=Z$, for $\lambda>0$. Thus, by Lumer–Phillips theorem, $\mathscr{A}$ generates a strongly continuous contraction semi-group on Z and the resolvent operator $R(\lambda,\mathscr{A})$ is compact on Z for* $\operatorname{Re}\lambda>0$.

(ii) *If Γ is of class C^1 or else Ω is a parallelepiped, then the spectrum $\sigma(\mathscr{A})$ of $\mathscr{A}$—which is only point spectrum by* (i)*—is contained in the open left plane* $\operatorname{Re}\lambda<0$: $\sigma(\mathscr{A})\subset\{\lambda: \operatorname{Re}\lambda<0\}$,

(iii) $D^*w_t\in L_2(0,\infty;L_2(\Gamma))$, *where* $w_t(t)\equiv w_t(t;w_0,w_1)$; *more precisely*

$$\int_0^\infty \|D^*w_t(t)\|_\Gamma^2\,dt\leqslant\left\|\begin{vmatrix}w_0\\w_1\end{vmatrix}\right\|^2\equiv E(0) \tag{1.19}$$

(iv) *For any $[w_0,w_1]\in Z$, we have*

$$\left\|\begin{vmatrix}w(t)\\w_t(t)\end{vmatrix}\right\|_Z=\left\|e^{\mathscr{A}t}\begin{vmatrix}w_0\\w_1\end{vmatrix}\right\|_Z\to 0, \qquad \text{as} \quad t\to+\infty \tag{1.20}$$

for the corresponding solution of the feedback system (1.18).

In order to obtain our main result on *uniform exponential* decay of the feedback semigroup $e^{\mathscr{A}t}$ on $\mathscr{L}(Z)$, and thus improve (1.20) from the strong to the uniform topology, we need further assumptions on the domain Ω.

VECTOR FIELD ASSUMPTION. *We assume that there exists a vector field* $h = [h_1(x),..., h_n(x)] \in C^2(\bar{\Omega})$ *defined on* $\bar{\Omega}$, *such that* (H1) *h is normal to* Γ, *at each point of* Γ; *i.e.,* $h(\sigma) \equiv k(\sigma)\nu$, $k(\sigma) = \text{scalar} \in C(\Gamma)$, $\nu = $ *unit outward normal* (H2) *for the transpose* $H(x)$ *of the Jacobian matrix of h*

$$H(x) = \begin{vmatrix} \dfrac{\partial h_1}{\partial x_1},..., \dfrac{\partial h_1}{\partial x_n} \\ \dfrac{\partial h_2}{\partial x_1},..., \dfrac{\partial h_2}{\partial x_n} \\ \vdots \qquad \vdots \\ \dfrac{\partial h_n}{\partial x_1},..., \dfrac{\partial h_n}{\partial x_n} \end{vmatrix} \tag{1.21a}$$

the following pointwise positive definiteness holds true: for any vector $v \in R^n$, *and* $x \in \bar{\Omega}$,

$$H(x)v \cdot v \geqslant \rho\ |v|_{R^n} \text{ for some } \rho > 0, \qquad \cdot = R^n\text{-inner product} \tag{1.21b}$$[1]

Actually, all we shall really need (following Eq. (3.58) *below) is the weaker integral version of* (1.21b): *for any* R^n*-vector* $v(x)$ *with* $|v(\cdot)|_{R^n} \in L_2(\Omega)$, *then*

$$\int_\Omega H(x)\, v(x) \cdot v(x)\, d\Omega \geqslant \rho \int_\Omega |v(x)|^2_{R^n}\, d\Omega \equiv \rho \|\, |v(x)|_{R^n} \|^2_{L_2(\Omega)} \quad \text{for some } \rho > 0. \tag{1.21c}$$

Our main theorem is then

THEOREM 1.2 (Uniform exponential stabilization). *Let the vector field assumption hold for* Ω. *Then, there exist constants* $C, \delta > 0$ *such that, for any* $[w_0, w_1] \in Z \equiv L_2(\Omega) \otimes H^{-1}(\Omega)$, *the corresponding solution of the feedback system* (1.1), (1.17) *satisfies*

$$\left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_Z \equiv \left\| e^{\mathscr{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z \leqslant Ce^{-\delta t} \left\| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z, \qquad t \geqslant 0$$

Domains Ω *Satisfying the Vector Field Assumption.* If Ω is a sphere (say, centered at the origin 0), or the set between two concentric spheres,

[1] The condition det $H(x) > 0$ on $\bar{\Omega}$ then follows from (1.21b).

then the vector field assumption is plainly satisfied by taking the radial vector field $h(P) = OP$.

In general, a full characterization of domains Ω which satisfy the vector field assumption appears to be unknown (and an interesting topic of research). We are most grateful to Professor Walter Littman, University of Minnesota, for a number of conversations on this topic; in particular, for pointing out to us that the vector field assumption is indeed satisfied for *strictly convex smooth domains* Ω, as a consequence (a fortiori!) of the main result in [CNS1] on the Monge–Ampere equation: Indeed, this result gives, in particular, a unique strictly convex $C^\infty(\bar{\Omega})$-solution of the problem: Hessian of $u \equiv 1$ in Ω, $u \equiv 0$ on Γ, for a strictly convex C^∞-domain Ω. The convexity of u then implies, upon setting $h = \nabla u$, that the matrix $H(x) =$ Hessian matrix of u be positive semi-definite. This, together with Hessian of $u \equiv 1$ on Ω, yields that $H(x)$ is positive definite on Ω.

Another proof that a strictly convex domain Ω of R^n with boundary of class C^2 satisfies the vector field assumption was shown to us by Professor Ennio De Giorgi, Scuola Normale Superiore, Pisa. This proof is short, completely elementary and very nice. Given a strictly convex closed set D of R^n with boundary of class C^2, De Giorgi *constructs* (see Appendix B) a strictly convex function $f(x)$ of class C^∞ in D such that: (i) $D = \{x \in R^n: 0 < f(x) \leqslant 1\}$, (ii) $f(\partial D) \equiv 1$, $\partial D =$ boundary of D. Hence, $h(x) \equiv \nabla f(x)$ satisfies the vector field assumption on D, with Jacobian matrix $(h) =$ Hessian matrix $(f) > 0$ on D. Similarly, the vector field assumption holds for sets of the type $D = D_1 \backslash D_2$, set difference of two strictly convex sets D_1 and D_2 of R^n, $\bar{D}_1 \nsubseteq D_2$, with C^2 boundary $\partial D = \partial D_1 \cup \partial D_2$, ∂D_1 and ∂D_2 disjoint. They can be described as $D = \{x \in R^n: k_1 \leqslant f(x) \leqslant k_2\}$ for a suitable strictly convex function $f(x)$ of class C^2, with $f(\partial D_1) \equiv k_1$ and $f(\partial D_2) \equiv k_2$. Then, $h(x) = \nabla f(x)$ satisfies the vector field assumption.

Literature

On the question of forcing all closed loop solutions of the wave equation on a bounded domain to decay to zero (in the strong or uniform topology), the only literature we are aware of, apart from [LT4] already quoted, concerns the case of feedback operators acting in the Neumann B.C. In contrast, no comparable results were available—to our knowledge—in the case of feedback operators acting in the Dirichlet B.C. The Neumann case with feedback $Fw_t = -w_t$, i.e., with B.C. $(\partial w/\partial \nu) = -w_t$ on $(0, \infty) \times \Gamma$ (as in our Sect. 4) has been studied by a number of authors, with decay achieved in the natural (energy) norm: $\| |\nabla w(t)| \|^2_\Omega + \|w_t(t)\|^2_\Omega$. Strong stability has been obtained, by means of different techniques, in [QR1] (by a compactness argument and Holmgren uniqueness theorem) and in [L1], while weak stability was obtained in [S2] (using La Salle's invariance principle in infinite dimensions). (A new operator theoretic proof for strong stability is presented in this paper, see proof of Theorem 1.1, and its adaptation to the Neumann case as explained at the beginning of Sect. 4). Results on strong stabilization are also contained in [Z1]. The much more demanding result on uniform (exponential) energy decay required new techniques, which were adapted from the literature of the so called "exterior" problem: energy decay of the wave equation exterior to a bounded obstacle Morewetz, Lax, Phillips, Ralston and Strauss]. By adapting these techniques to the case of a bounded domain, uniform exponential energy decay was first obtained in [C1–2] for certain classes of domains, and later extended in [L1] to a much greater generality and completeness. [Strong stability results of the wave equation were also obtained in [LT5] with Neumann B.C. $(\partial w(t)/\partial \nu) = (w_t(t)|_\Gamma, g_1)_\Gamma\, g_2$ on $(0, \infty) \times \Gamma$, for large classes of boundary vectors, including—but not restricted to—the dissipative case.]

As to the case of feedback operators acting in the Dirichlet B.C., no prior result was available for exponential (uniform) stability. Our guiding reference for the present paper is [L1] and we make crucial use in Proposition 3.1 of some estimates by adapting a multiplier technique as in [LLT1]. In Section 4, we provide a sketch of the same techniques as applied to the Neumann B.C. $(\partial w/\partial \nu) = -w_t$ on $(0, \infty) \times \Gamma$, thereby reproving, under slightly weaker conditions on Ω, the main result of [L1] on exponential decay of the energy. (We use (1.21c) instead of (1.21b)).

Remarks on Exact Controllability for (1.1). It is by now well known that there exists a relationship between the existence of a uniform decay rate for the closed loop hyperbolic problem (as asserted by our Theorem 1.2 on the space Z) and the corresponding exact controllability problem for the open loop problem (1.1). More precisely, the uniform operator decay rate on the space $Z = L_2(\Omega) \otimes H^{-1}(\Omega)$ for the closed loop problem (1.1), (1.17) implies (null controllability, hence, by time reversability) exact controllability on a finite, sufficiently large (universal) time T within the space Z for the corresponding open loop system (1.1), with $L_2(0, T; L_2(\Gamma))$-Dirichlet boundary control. This, and more, follows from the "controllability via stabilizability" argument first pointed out by Russell in [R1], to which we refer for details. Illustrations of this method (in the case of the wave equation with Neumann B.C.) are given in [C1], etc. This way, we arrive at the following new theorem, which is topologically charper than previously known results with Dirichlet boundary control, as in [R1, 2, L5, 6] etc. in that our result is claimed for the first time in the natural ([L4, LT2, LLT1]) state space Z and not, as in previous literature, on smooth subspaces of it.

THEOREM 1.3 (exact boundary controllability on finite time $T < \infty$ on the state space $Z = L_2(\Omega) \times H^{-1}(\Omega)$). *Let the conclusion on uniform decay on Z as in Theorem* 1.2 *hold. Then, given any initial data* $(w_0, w_1) \in Z$ *and any target data* $(v_0, v_1) \in Z$, *there exists a boundary control* $u \in L_2(0, T; L_2(\Gamma))$, *for T finite, sufficiently large and independent on initial and target data, such that once inserted as a Dirichlet control in* (1.1c) *produces a solution w of the corresponding problem* (1.1) *which satisfies* $[w(t), w_t(t)] \in C([0, T]; Z)$ *and* $w(T) = v_0$, $w_t(T) = v_1$. *Moreover,*

$$\|u\|_{L_2(0,T;L_2(\Gamma))} \leqslant C_T \left\{ \left\| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z + \left\| \begin{vmatrix} v_0 \\ v_1 \end{vmatrix} \right\|_Z \right\}.$$

Remark 1.1. The converse of the "controllability via stabilizability" implication also holds true. Let (1.1) be null-controllable on Z in finite time $T < \infty$; i.e., any pair $(w_0, w_1) \in Z$ of initial data can be steered to the zero state $(0, 0)$ over the finite interval $[0, T]$, by means of an $L_2(0, T; L_2(\Gamma))$-Dirichlet control u. Then, extending such u by zero after time T, yields that the "finite cost condition" of the corresponding regulator problem is satisfied (see comments on point (ii) of our motivation below (1.4)). Hence, as explained there, the Riccati theory developed in [LT3, Sect. 5] yields the feedback operator $u^0(t) = B^*P \left|\begin{smallmatrix} w^0(t) \\ w_t^0(t) \end{smallmatrix}\right| \in L_2(0, \infty; L_2(\Gamma))$ which produces exponential decay in the uniform operator topology of Z of the semi-group of the corresponding closed loop system. Thus, uniform exponential decay in Z (Theorem 1.2) and exact controllability on Z in finite time (Theorem 1.3) are equivalent properties.

2. Preliminary Results and Proof of Theorem 1.1

We begin by collecing some properties of the operator $\mathscr{A}$ defined in (1.12), (1.14): this will amount to proving parts (i) and (ii) of Theorem 1.1.

Lemma 2.1. (i) *The operator $\mathscr{A}$ in* (1.12), (1.14) *is dissipative on* $Z = L_2(\Omega) \otimes H^{-1}(\Omega)$ *and satisfies here*: *range of* $(\lambda I - \mathscr{A}) = Z$, *for* $\lambda > 0$. *Thus, by Lumer–Phillips theorem, $\mathscr{A}$ generates a* contraction s.c. *semigroup on* $Z = L_2(\Omega) \times H^{-1}(\Omega)$:

$$e^{\mathscr{A} t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} = \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \tag{2.1}$$

(ii) *The resolvent operator $R(\lambda, \mathscr{A})$ of $\mathscr{A}$ is given by*

$$R(\lambda, \mathscr{A}) = \begin{vmatrix} V(\lambda)[\lambda A^{-1} + DD^*] & V(\lambda) A^{-1} \\ \lambda V(\lambda)[\lambda A^{-1} + DD^*] - I & \lambda V(\lambda) A^{-1} \end{vmatrix} \tag{2.2}$$

where we have set

$$V(\lambda) = [I + \lambda DD^* + \lambda^2 A^{-1}]^{-1} \tag{2.3}$$

for at least all λ *with* $\operatorname{Re}\lambda > 0$; *moreover,* $R(\lambda, \mathscr{A})$ *is compact as an operator on* Z.

(iii) *When* Γ *is of class* C^1 *or else* Ω *is a parallelepiped,* $R(\lambda, \mathscr{A})$ *is well defined (and compact) on* Z, *on the closed half plane* $\operatorname{Re}\lambda \geqslant 0$. *Consequently, the spectrum (point spectrum)* $\sigma(\mathscr{A})$ *of* $\mathscr{A}$ *satisfies*

$$\sigma(\mathscr{A}) \subset \{\lambda: \operatorname{Re}\lambda < 0\}. \tag{2.4}$$

Proof. (i) Dissipativity of $\mathscr{A}$ on Z was already verified in (1.13). Next, with $\lambda > 0$ fixed, let $z \in Z$ and solve $(\lambda I - \mathscr{A})x = z$, i.e.,

$$\begin{aligned} \lambda x_1 - x_2 &= z_1 \in L_2(\Omega) && \text{(a)} \\ Ax_1 + \lambda x_2 + ADD^* x_2 &= z_2 \in H^{-1}(\Omega) && \text{(b)} \end{aligned} \tag{2.5}$$

for $x \in \mathscr{D}(\mathscr{A})$. By substitution into the second equation and applying $A^{-1/2}$ to both sides, we obtain

$$\begin{aligned} &[I + \lambda DD^* + \lambda^2 A^{-1}]x_1 = y \\ &\qquad y = A^{-1}z_2 + (\lambda A^{-1} + DD^*)z_1 \in H^1(\Omega). \end{aligned} \tag{2.6}$$

Since the strictly positive definite operator within brackets is boundedly invertible on $L_2(\Omega)$, we have that

$$\begin{aligned} x_1 &= [I + \lambda DD^* + \lambda^2 A^{-1}]^{-1} y \\ x_2 &= \lambda x_1 - z_1 \end{aligned} \tag{2.7}$$

is the unique solution of (2.5). Plainly, $x_i \in L_2(\Omega)$. Actually, since $A^{-1}z_2 \in \mathscr{D}(A^{1/2})$ (see (2.5b)), we obtain from (2.6)

$$\begin{aligned} x_1 + DD^* x_2 &= x_1 + DD^*(\lambda x_1 - z_1) \\ &= -\lambda^2 A^{-1} x_1 + A^{-1} z_2 + \lambda A^{-1} z_1 \in \mathscr{D}(A^{1/2}) \end{aligned}$$

and $[x_1, x_2] \in \mathscr{D}(\mathscr{A})$ (see (1.14)). Part (i) is proved. Re-writing (2.7) explicitly yields the expression (2.2) for $R(\lambda, \mathscr{A})$, from which compactness follows at once. It remains to show that the spectrum of $\mathscr{A}$—which so far, by part (i), is contained in $\{\lambda: \operatorname{Re}\lambda \leqslant 0\}$—does not contain the imaginary axis. This will follow from showing that the operator $V(\lambda)$, which is a well-defined bounded operator in $L_2(\Omega)$ for $|\lambda|$ suitably small, remains so also for $\lambda = ir$, r is real and say $\neq 0$. Letting

$$[I + (ir)DD^* + (ir)^2 A^{-1}]x = 0, \; x \in L_2(\Omega) \tag{2.8}$$

we show that $x=0$. Indeed taking inner product with x

$$\|x\|_\Omega^2 - r^2(A^{-1}x, x) + ir\,\|D^*x\|_\Gamma^2 = 0 \tag{2.9}$$

with $\|x\|_\Omega^2 - r^2(A^{-1}x, x) = \text{real}$, hence

$$D^*x = 0. \tag{2.10}$$

Returning to (2.8) we get

$$Ax = r^2 x.$$

Thus, either $x=0$ and we are done, or else x is an eigenvector of A, say $x=e_n$, with corresponding eigenvalue $\mu_n = r^2$. Returning to (2.10), we then have $D^*e_n = 0$, i.e., by (1.10) $(\partial/\partial\nu)(A^{-1}e_n)|_\Gamma = (1/\mu_n)(\partial e_n/\partial\nu)|_\Gamma = 0$. But, under the assumptions made on Γ, we have that $e_n|_\Gamma = 0$ and $(\partial e_n/\partial\nu)|_\Gamma = 0$ imply $e_n = 0$, i.e., $x=0$ and $I + \lambda DD^* + \lambda^2 A^{-1}$ is injective for $\lambda = ir$. Since $\lambda DD^* + \lambda^2 A^{-1}$ is compact in $L_2(\Omega)$, see (1.15), the inverse $V(\lambda)$ in (2.3) exists as a bounded operator also on the imaginary axis. Inclusion (2.4) is proved. As a corollary of (1.13) we obtain part (iii) of Theorem 1.1. ▮

COROLLARY 2.2. *With $w_t(t) = w_t(t; w_0, w_1)$, the map D^*w_t: continuous $[w_0, w_1] \in Z \to L_2(0, \infty; L_2(\Gamma))$, more precisely*

$$\int_0^\infty \|D^*w_t(t)\|_\Gamma^2\,dt \leqslant \left\| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z^2.$$

Proof. From (1.13) with $y = e^{\mathscr{A}t}\left|\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right|$, $\left|\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right| \in \mathscr{D}(\mathscr{A})$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\| e^{\mathscr{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z^2 &= \frac{1}{2}\frac{d}{dt}\left(e^{\mathscr{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}, e^{\mathscr{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right)_Z \\
&= \operatorname{Re}\left(\mathscr{A}e^{\mathscr{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}, e^{\mathscr{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right)_Z \\
&= \operatorname{Re}\left(\mathscr{A}\begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix}, \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right)_Z = -\|D^*w_t(t)\|_\Gamma^2
\end{aligned}$$

from which

$$\int_0^\infty \|D^*w_t(t)\|_\Gamma^2\,dt = -\frac{1}{2}\lim_{t\uparrow\infty}\left\| e^{\mathscr{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z^2 + \frac{1}{2}\left\| \begin{matrix} w_0 \\ w_1 \end{matrix} \right\|_Z^2 \leqslant \left\| \begin{matrix} w_0 \\ w_1 \end{matrix} \right\|_Z^2 < \infty \tag{2.11}$$

by contraction of $e^{\mathscr{A}t}$ on Z. Extension by continuity of (2.11) to all $[w_0, w_1] \in Z$ yields the conclusion. ▮

Remark 2.1. For $\Gamma \in C^1$ or else $\Omega =$ parallelepiped, the next result will show, as a consequence of Lemma 2.1(ii) that, in fact,

$$\left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_Z = \left\| e^{\mathscr{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z \to 0 \text{ as } t \to \infty$$

for all $[w_0, w_1] \in Z$. Thus, Corollary 2.2 can be refined to read

$$\int_0^\infty \|D^* w_t(t)\|_\Gamma^2 \, dt \leqslant \frac{1}{2} \left\| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z$$

We now pass to part (iv) of Theorem 1.1.

Proof of Theorem 1.1(iv) (*strong stability*). To complete the proof of strong stability, we may proceed, e.g., as in [LT4]. Since $e^{\mathscr{A}t}$ is, by Lemma 2.1, a s.c. contraction semigroup on Z, the Nagy–Foias–Fogel decomposition theory applies for it. (For an excellent expository treatment of this theory, as applied to stabilization problems, see [L3]). Accordingly, Z can be decomposed in a unique way into the orthogonal sum of three subspaces Z_{cnu}, W_u and $W^\perp$, all reducing for $e^{\mathscr{A}t}$ and its adjoint

$$Z = Z_{cnu} + W_u + W^\perp$$

such that

$$W_u \oplus W^\perp = Z_u \qquad W_u \oplus Z_{cnu} = W$$

where

(i) on Z_{cnu}: $e^{\mathscr{A}t}$ is completely nonunitary and weekly stable;

(ii) on Z_u: $e^{\mathscr{A}t}$ is a unitary s.c. group. It follows that in our present case, $Z_u = \{0\}$, the trivial subspace, for otherwise Stone's theorem would yield that the eigenvalues of $\mathscr{A}$ on Z_u are on the imaginary axis, and this would contradict Lemma 2.1(ii) when $\Gamma \in C^1$ or else $\Omega =$ parallelepiped. Thus, in our case, $Z = Z_{cnu}$ and $e^{\mathscr{A}t}$ is weakly stable on Z; however, since $\mathscr{A}$ has compact resolvent on Z by Lemma 2.1, then $e^{\mathscr{A}t}$ is actually stable in the strong topology of Z $[B \cdot 1]$: $e^{\mathscr{A}t}z \to 0$, $z \in Z$. Theorem 1.1 is fully proved. ∎

3. Proof of Main Theorem 1.2

3.1 *Preliminary Results and Outline of Strategy*

For the feedback problem (1.8) we define the ("energy") $E(w, t)$ by the squared norm of the semigroup in (2.1)

$$E(t) \equiv E(w, t) = \left\| e^{\mathscr{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z^2 = \left\| \begin{matrix} w(t) \\ w_t(t) \end{matrix} \right\|_Z^2 = \|w(t)\|_\Omega^2 + \|A^{-1/2} w_t(t)\|_\Omega^2$$

$$\leqslant E(t_0), \; t \geqslant t_0 \tag{3.1}$$

by the contraction property in Lemma 2.1, where $w(t) \equiv w(t; w_0, w_1)$, $w_t(t) \equiv w_t(t; w_0, w_1)$, $[w_0, w_1] \in Z$.

Following, in part, [L1], we introduce

$$Q(t) = Q_1(t) + Q_2(t) \tag{3.2}$$

$$Q_1(t) = qtE(t) \qquad \text{(a)}$$

$$Q_2(t) = \tfrac{1}{2}((r - \operatorname{div} h)\, A^{-1} w_t(t), w(t))_\Omega \qquad \text{(b)} \tag{3.3}$$

Here, q is a positive constant, r another constant, both to be chosen at the end of the proof below (3.69), while $h(x) = [h_1(x), ..., h_n(x)]$ is—for the time being—a smooth, time independent vector field on $\bar{\Omega}$, say $h \in C^2(\bar{\Omega})$. Schwarz inequality on (3.3b) yields

$$Q_2(t) \leqslant C_{h,r} \, \|A^{-1/2}\| \, \{ \|A^{-1/2} w_t(t)\|_\Omega^2 + \|w(t)\|_\Omega^2 \} \leqslant \mathrm{Const}_{h,r} E(t_0), \qquad t \geqslant t_0$$

so that, for any constant $\beta > 0$, we conclude from here and (3.1)–(3.3)

$$\lim_{t \to \infty} e^{-\beta t} Q(t) = 0, \tag{3.4}$$

a result needed below. From (3.3a), (3.1) we compute

$$\frac{dQ_1(t)}{dt} = qE(t) + 2qt[w_t(t), w(t))_\Omega + (A^{-1/2} w_{tt}(t), A^{-1/2} w_t(t))_\Omega]$$

$$\text{(by (1.16))} = qE(t) + 2qt[(w_t(t), w(t))_\Omega - (A^{-1/2} w(t), A^{-1/2} w(t))_\Omega$$
$$- (D^* w_t(t), D^* w_t(t))_\Gamma].$$

Recalling Corollary 2.2,

$$\frac{dQ_1(t)}{dt} = qE(t) - 2qt \, \|D^* w_t(t)\|_\Gamma^2 \in L_2(0, T), \qquad \text{any} \quad T < \infty. \tag{3.5}$$

Similarly, from (3.3b) and (1.16)

$$\frac{dQ_2(t)}{dt} = \frac{1}{2} (\operatorname{div} h - r)(w(t) + DD^* w_t(t), w(t))_\Omega$$

$$+ \frac{1}{2} ((r - \operatorname{div} h)\, A^{-1} w_t(t), w_t(t))_\Omega$$

$$= \frac{r}{2} \|A^{-1/2} w_t(t)\|_\Omega^2 - \frac{r}{2} \|w(t)\|_\Omega^2 - \frac{r}{2} (DD^* w_t(t), w(t))_\Omega$$

$$+ \frac{1}{2} (w(t) \operatorname{div} h, w(t))_\Omega + \frac{1}{2} (DD^* w_t(t) \operatorname{div} h, w(t))_\Omega$$

$$- \frac{1}{2} (A^{-1} w_t(t) \operatorname{div} h, w_t(t))_\Omega \in L_2(0, T), \qquad \text{any} \quad T < \infty \tag{3.6}$$

again by Corollary 2.2. Thus, for any constant $\beta > 0$ and some $t_0 > 0$ to be further specified below (3.69), we compute from (3.2), (3.5)–(3.6)

$$\int_{t_0}^{\infty} e^{-2\beta t} \frac{dQ(t)}{dt}\, dt = q \int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt - 2q \int_{t_0}^{\infty} t e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2\, dt$$
$$+ \frac{r}{2} \int_{t_0}^{\infty} e^{-2\beta t} [\|A^{-1/2} w_t(t)\|_\Omega^2 - \|w(t)\|_\Omega^2]\, dt$$
$$+ \int_{t_0}^{\infty} e^{-2\beta t} \left[\frac{1}{2} (DD^* w_t(t) \operatorname{div} h, w(t))_\Omega \right.$$
$$\left. - \frac{r}{2} (DD^* w_t(t), w(t))_\Omega \right] dt$$
$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} [(w(t) \operatorname{div} h, w(t))_\Omega$$
$$- (A^{-1} w_t(t) \operatorname{div} h, w_t(t))_\Omega]\, dt \tag{3.7}$$

with the right-hand side of (3.7) finite for any $\beta > 0$, by Lemma 2.1(i) and Corollary 2.2. Our *goal* is then to show that, for a suitable choice of the constants $q > 0$ and r, the right-hand side (R.H.S.) of (3.7) satisfies the inequality

$$\text{R.H.S. of (3.7)} \leqslant -C_{q,r,t_0}^2 \int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt + C_{q,r,t_0}'^2 E(t_0) \tag{3.8}$$

for some $t_0 > 0$, with (positive) constants C_{q,r,t_0}^2, $C_{q,r,t_0}'^2 E(t_0)$ depending on q, r, t_0, *but independent* of $\beta > 0$, and for all initial data $[w(t_0), w_t(t_0)] \in Z$. Indeed, since

$$\int_{t_0}^{\infty} e^{-2\beta t} \frac{dQ}{dt}(t)\, dt = 2\beta \int_{t_0}^{\infty} e^{-2\beta t} Q(t)\, dt - e^{-2\beta t_0} Q(t_0) \tag{3.9}$$

by integration by parts with the help of (3.4), we see from (3.7)–(3.9) that then

$$2\beta \int_{t_0}^{\infty} e^{-2\beta t} Q(t)\, dt + C_{q,r,t_0}^2 \int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt$$
$$\leqslant C_{q,r,t_0}'^2 E(t_0) + e^{-2\beta t_0} [q t_0 + c_{r,h}]\, E(t_0) \tag{3.10}$$

after estimating $Q(t_0)$ via (3.2)–(3.3). Moreover, since $Q(t) \geqslant 0$ for $t \geqslant t_0$ with t_0 sufficiently large, we may drop the first term in (3.10), thus obtaining

$$\int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt \leqslant \text{const}_{q,r,t_0} E(t_0), \qquad \text{for all } \beta > 0, \text{ all } [w(t_0), w_t(t_0)] \in Z \tag{3.11a}$$

where const does *not* depend on $\beta > 0$. Letting $\beta \downarrow 0$ and using the contraction property of $E(t)$, we plainly conclude that

$$\int_0^\infty E(t)\,dt \leqslant \text{const}\, E(0), \qquad [w_0, w_1] \in Z \tag{3.11b}$$

where const is *independent* on the initial data $[w_0, w_1] \in Z$. Thus, for the semigroup $e^{\mathscr{A}t}$ in (3.1), a well-known result [D1] applies under condition (3.11): there are positive constants δ, M_δ such that

$$\sqrt{E(t)} = \left\| e^{\mathscr{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z \leqslant M_\delta\, e^{-\delta t} \left\| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z, \qquad t \geqslant 0, \delta > 0 \tag{3.12}$$

$[w_0, w_1] \in Z$ and our desired conclusion is achieved. Thus, our Main Theorem *is proved, as soon as we establish inequality)* (3.8), *under the assumptions* of Section 1 *on the vector field h* on $\bar{\Omega}$. To this end, with reference to the R.H.S. of (3.7), we set for convenience

$$G_1(w) \equiv G_{1,q,r,t_0,\beta}(t)$$
$$= q \int_{t_0}^\infty e^{-2\beta t} E(t)\,dt + \frac{r}{2} \int_{t_0}^\infty e^{-2\beta t} [\|A^{-1/2} w_t(t)\|_\Omega^2 - \|w(t)\|_\Omega^2]\,dt \tag{3.13}$$

$$G_2(w) \equiv G_{2,r,t_0,\beta}(t)$$
$$= \int_{t_0}^\infty e^{-2\beta t} \left[\frac{1}{2} (DD^* w_t(t) \operatorname{div} h, w(t))_\Omega - \frac{r}{2} (DD^* w_t(t), w(t))_\Omega \right] dt \tag{3.14}$$

$$+ \tfrac{1}{2} \int_{t_0}^\infty e^{-2\beta t} [(w(t) \operatorname{div} h, w(t))_\Omega - (A^{-1} w_t(t) \operatorname{div} h, w_t(t))_\Omega]\,dt \tag{3.15}$$

whereby (3.7) is more concisely rewritten as

$$\int_{t_0}^\infty e^{-2\beta t} \frac{dQ}{dt}(t)\,dt = G_1(w) + G_2(w) - 2q \int_{t_0}^\infty t e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2\,dt. \tag{3.7'}$$

Then, plainly, inequality (3.8) holds true a fortiori, *if we can establish* that for suitable constants $q > 0$, r and $t_0 > 0$, and for all $\beta > 0$, we have

$$G_1(w) + G_2(w) - 2q t_0 \int_{t_0}^\infty e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2\,dt$$
$$\leqslant -C_{q,r,t_0}^2 \int_{t_0}^\infty e^{-2\beta t} E(t)\,dt + C_{q,r,t_0}'^2 E(t_0) \tag{3.16}$$

for all initial data $[w(t_0), w_t(t_0)] \in Z$.

Orientation

Since the proof of inequality (3.16) is lengthly and technical, we outline here at the outset the guiding strategy used in establishing it. We shall need to compare the energy terms

$$\int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|_\Omega^2 \, dt, \int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_\Omega^2 \, dt \qquad (\#)$$

on the one hand, with the boundary dissipation term

$$\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2 \, dt \qquad (\#\#)$$

on the other. More precisely, we shall seek to express $G_2(w)$ in (3.15) as the sum of terms of the following three kinds:

(i) whenever possible, either terms as in (#) premultiplied by a *negative* coefficient, or else terms $\mathcal{O}(E(t_0))$ (by convention, $\mathcal{O}$ will mean bounded by a constant *independent of* β or t_0)

(ii) if unable to achieve goal (i), then terms as in (#) pre-multiplied by a *positive* coefficient which we then must endeavor to make *arbitrarily small*;

(iii) terms as in (# #), even if premultiplied by positive coefficients, *however independent on* t_0.

The ultimate goal is then to select suitable constants $q > 0$, r such that

[terms in (i) and (ii)] $+ G_1(w) \leq -C_{q,r,t_0}^2 \int_{t_0}^{\infty} e^{-2\beta t} E(t) \, dt + \mathcal{O}(E(t_0))$

for all $\beta > 0$

while the terms in (iii), having (possibly positive) coefficients independent on t_0, are then compensated by the remaining term $-2qt_0 \int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2 \, dt$ in (3.16), for a suitable choice of $t_0 > 0$ sufficiently large.

The program regarding the decomposition of $G_2(t)$ in terms of the type described in (i), (ii), (iii) is achieved in Propositions 3.6 and 3.7 and culminates in relation (3.68). As a preliminary step toward this decomposition, we easily obtain from (3.15) by virtue of D being bounded $L_2(\Gamma) \to L_2(\Omega)$ (see (1.6)),

$$\begin{aligned} G_2(w) = \tfrac{1}{2} \int_{t_0}^{\infty} & e^{-2\beta t} [(w(t) \operatorname{div} h, w(t))_\Omega - (A^{-1} w_t(t) \operatorname{div} h, w_t(t))_\Omega] \, dt \\ & + (1 + |r|) \left\{ \varepsilon_1 \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_\Omega^2 \, dt \right) \right. \\ & \left. + \frac{1}{\varepsilon_1} \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2 \, dt \right) \right\} \end{aligned} \qquad (3.17)$$

where $\mathcal{O}$ denotes upper bound depending on $\max_{\Omega} |\operatorname{div} h|$, $\|D\|_{\mathscr{L}(L_2(\Gamma), L_2(\Omega))}$, in particular *independent* of $\beta > 0$ or $t_0 > 0$, and where $\varepsilon_1 > 0$ is arbitrary. Thus, according to the preceeding paragraph, we only need to obtain the desired decomposition for the first two terms of $G_2(w)$ given by (3.17). This will be accomplished in Propositions 3.6 and 3.7. Key to this end is the observation that, by (1.10), $D^* w_t = D^* A A^{-1} w_t = -(\partial/\partial \nu) A^{-1} w_t$, i.e., as normal derivative $\partial p/\partial \nu$ of a new variable $p = A^{-1} w_t$.

3.2. *An a Priori Identity*

It will suffice to establish the desired inequality (3.11b) for all initial data $[w_0, w_1] \in \mathscr{D}(\mathscr{A})$, see (1.14), and then extend by continuity to all $[w_0, w_1]$ Z, as usual. Thus, since inequality (3.16) implies [inequality (3.11a), hence] inequality (3.11b), it will suffice to establish (3.16) under the *assumption—kept henceforth unless otherwise stated*—that $[w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A})$. Recalling (1.14), we see that then the corresponding solution satisfies

$$w(t) = w(t; w(t_0), w_t(t_0)) \in C([t_0, T]; H^1(\Omega)) \tag{3.18a}$$

$$[w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A})$$

$$w_t(t) = w_t(t; w(t_0), w_t(t_0)) \in C([t_0, T]; L_2(\Omega)), \tag{3.18b}$$

$$w_{tt}(t) = w_{tt}(t; w(t_0), w_t(t_0)) \in C([t_0, T]; H^{-1}(\Omega)) \tag{3.18c}$$

for any $t_0 < T < \infty$. The left-hand side of (3.16) contains the boundary term $\|D^* w_t(t)\|_\Gamma$, see Corollary 2.2. By (1.10), this suggests introducing a new variable $p(t)$:

$$D^* w_t(t) = D^* A A^{-1} w_t(t) = -\frac{\partial}{\partial \nu} p(t) \qquad \text{on } [t_0, T] \times \Gamma \tag{3.19}$$

$$p(t) \equiv A^{-1} w_t(t) \begin{cases} \in C([t_0, T], \mathscr{D}(A^{1/2})), & [w(t_0), w_t(t_0)] \in Z \\ \in C([t_0, T]; \mathscr{D}(A)), & [w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A}) \end{cases} \tag{3.20}$$

by Lemma 2.1 and (3.18b). Moreover, by (1.6), Lemma 2.1 and Corollary 2.2 we get the top inclusion of

$$p_t(t) = A^{-1} w_{tt}(t) = -[w(t) + DD^* w_t(t)]$$

$$\times \begin{cases} \in L_2(t_0, T; L_2(\Omega)), & \text{for } [w_0, w_1] \in Z \quad \text{(a)} \\ \in C([t_0, T]; H^1(\Omega)), & \text{for } [w_0, w_1] \in \mathscr{D}(\mathscr{A}) \quad \text{(b)} \end{cases} \tag{3.21}$$

while the inclusion at the bottom follows by (3.18) and $DD^*\colon L_2(\Omega) \to H^1(\Omega)$ continuously, see (1.15). Hence,

$$p_{tt}(t) = -w_t(t) - DD^* w_{tt}(t) = -Ap(t) - DD^* w_{tt}(t). \tag{3.22}$$

In terms of the scalar function $p(t, x)$, $x \in \Omega$, corresponding to the vector valued function $p(t) = p(t, \cdot)$, Eqs. (3.20)–(3.22) are written explicitly as the following hyperbolic problem

$$\frac{\partial^2 p}{\partial t^2} = \Delta p + F \qquad \text{on } (t_0, \infty) \times \Omega \quad \text{(a)}$$

$$p(t_0, \cdot) = A^{-1} w_t(t_0),\ p_t(t_0, \cdot) = -[w(t_0) + DD^* w_t(t_0)] \quad \text{in } \Omega \qquad \text{(b)}$$

$$p = 0 \qquad \text{in } (t_0, \infty) \times \Gamma \quad \text{(c)} \tag{3.23}$$

(the homogeneous Dirichlet B.C. on p stems from $p(t) \in \mathscr{D}(A)$), where

$$F = -DD^* w_{tt}. \tag{3.24}$$

In our argument, we shall have to consider the pointwise values $p_t(t_0)$, $p_t(T)$, etc. in $L_2(\Omega)$: notice from (3.21b) that they make sense for initial data $[w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A})$, *as assumed*, while from (3.21a) the pointwise meaning for $p_t(t)$ in $L_2(\Omega)$ is lost for general initial data in Z. By (3.20)

$$A^{-1/2} w_t(t) = A^{1/2} p(t) \begin{cases} \in C([t_0, T]; L_2(\Omega)), & [w(t_0), w_t(t_0)] \in Z \quad \text{(3.25a)} \\ \in C([t_0, T]; \mathscr{D}(A^{1/2})), & [w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A}) \end{cases} \tag{3.25b}$$

and by (3.25a) the following expressions are well defined for all t

$$\|A^{-1/2} w_t(t)\|_\Omega^2 = \|A^{1/2} p(t)\|_\Omega^2 = \| |\nabla p(t)| \|_\Omega^2 = \int_\Omega |\nabla p(t)|^2 \, d\Omega$$

$$\text{equivalent to } \|p(t)\|^2_{H_0^1(\Omega)}$$

$$\leqslant E(t) \leqslant E(t_0), \qquad t \geqslant t_0 \tag{3.26}$$

Since $\lambda = 0 \in \rho(\mathscr{A})$, the resolvent set of $\mathscr{A}$ (a special case contained in Lemma 2.1), then for $[w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A})$, and $\tau = t - t_0$:

$$\left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_{\mathscr{D}(\mathscr{A})} = \left\| e^{\mathscr{A}\tau} \begin{vmatrix} w(t_0) \\ w_t(t_0) \end{vmatrix} \right\|_{\mathscr{D}(\mathscr{A})} = \left\| e^{\mathscr{A}\tau} \begin{vmatrix} w(t_0) \\ w_t(t_0) \end{vmatrix} \right\|_Z$$

$$= \left\| e^{\mathscr{A}\tau} \mathscr{A} \begin{vmatrix} w(t_0) \\ w_t(t_0) \end{vmatrix} \right\|_Z \leqslant \left\| \mathscr{A} \begin{vmatrix} w(t_0) \\ w_t(t_0) \end{vmatrix} \right\|_Z, \qquad t \geqslant t_0 \tag{3.27a}$$

by the contraction property. A fortiori, (3.27a) implies via (1.14) and below that

$$\|w(t)\|^2_{H^1(\Omega)} + \|w_t(t)\|^2_\Omega \leqslant \left\| \mathscr{A} \begin{vmatrix} w(t_0) \\ w_t(t_0) \end{vmatrix} \right\|_Z^2 \qquad \text{all} \quad t \geqslant t_0. \tag{3.27b}$$

Thus, for $[w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A})$ and $T > t_0$ we obtain from (3.26), (3.21), and (3.27) using the contraction of $E(t)$

$$\| |\nabla p(T)| \|_\Omega^2 + \|p_t(T)\|_\Omega^2 \leqslant C\{E(T) + \|w(T)\|_\Omega^2 + \|w_t(T)\|_\Omega^2\}$$
$$\leqslant C\left\{E(t_0) + \left\| \mathscr{A} \begin{vmatrix} w(t_0) \\ w_t(t_0) \end{vmatrix} \right\|_Z^2 \right\} \tag{3.28}$$

uniformly in $T \geqslant t_0$, a result to be repeatedly invoked: here C depends on $1 + \|DD^*\|_{\mathscr{L}(L_2(\Omega))}$ and the constant of equivalence in (3.26). After the preliminaries, we pass now to analyze (see (3.19))

$$\int_{t_0}^\infty e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2 \, dt = \int_{t_0}^\infty e^{-2\beta t} \left\| \frac{\partial p}{\partial \nu}(t) \right\|_\Gamma^2 dt, \qquad \beta > 0$$

for problem (3.23). This is accomplished as a consequence of the following result, obtained by adapting to present circumstances the "multiplier" technique as in [LLT1].

PROPOSITION 3.1. *Let Ω possess a vector field $h(x) = [h_1(x), ..., h_n(x)] \in C^2(\bar{\Omega})$. Then, the following identity holds for problem* (3.23)–(3.24) *with $[w(t_0), w_t(t_0)] \in Z$ and for any $\beta > 0$*

$$\int_{t_0}^\infty e^{-2\beta t} \int_\Gamma \left[\frac{\partial p}{\partial \nu}(h \cdot \nabla p) - \frac{1}{2}\left(\frac{\partial p}{\partial \nu}\right)^2 h \cdot \nu\right] d\Gamma \, dt$$
$$= \int_{t_0}^\infty e^{-2\beta t} \int_\Omega (H\nabla p) \cdot \nabla p \, d\Omega \, dt - \frac{1}{2}\int_{t_0}^\infty e^{-2\beta t} \int_\Omega |\nabla p|^2 \operatorname{div} h \, d\Omega \, dt$$
$$+ \frac{1}{2}\int_{t_0}^\infty e^{-2\beta t} \int_\Omega p_t^2 \operatorname{div} h \, d\Omega \, dt - 2\beta \int_{t_0}^\infty e^{-2\beta t} \int_\Omega w(h \cdot \nabla p) \, d\Omega \, dt$$
$$+ e^{-2\beta t_0}(w(t_0), h \cdot A^{-1/2} w_t(t_0))_\Omega - \int_{t_0}^\infty e^{-2\beta t}(DD^* w_t, h \cdot \nabla p_t)_\Omega \, dt \tag{3.29}$$

where $H(x)$ is the $n \times n$ Jacobian matrix of $h(x)$

$$H(x) \equiv \begin{vmatrix} \dfrac{\partial h_1}{\partial x_1}, ..., \dfrac{\partial h_1}{\partial x_n} \\ \dfrac{\partial h_2}{\partial x_1}, ..., \dfrac{\partial h_2}{\partial x_n} \\ \vdots \qquad \vdots \\ \dfrac{\partial h_n}{\partial x_1}, ..., \dfrac{\partial h_n}{\partial x_n} \end{vmatrix}$$

and $H\nabla p$ denotes multiplication of the matrix H and the vector ∇p.

Remark 3.1. Before proving Theorem 3.1, we analyze the terms at the right of identity (3.29). From (3.26), (3.21), and Corollary 2.2, we can see that for $\beta > 0$ the integrals

$$\int_{t_0}^{\infty} e^{-2\beta t} \| |\nabla p(t)| \|_{\Omega}^2 \, dt; \int_{t_0}^{\infty} e^{-2\beta t} \| p_t(t) \|_{\Omega}^2 \, dt, \int_{t_0}^{\infty} e^{-2\beta t} \| DD^* w_t(t) \|_{\Omega}^2 \, dt$$

are all well defined (finite) for $[w(t_0), w_t(t_0)] \in Z$, using the contraction property of $E(t)$. Thus, all terms on the right of identity (3.29) are well defined, except possibly for the last term. That this last term is also well defined follows from the next Proposition.

PROPOSITION 3.2. *Under the vector field assumption, we have for $\beta > 0$ and $[w(t_0), w_t(t_0)]$*

$$\int_{t_0}^{\infty} e^{-2\beta t} (DD^* w_t(t), h \cdot \nabla p_t(t))_{\Omega} \, dt$$

$$= \mathcal{O}(E(t_0)) + \beta \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \| w(t) \|_{\Omega}^2 \, dt \right) + \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \| D^* w_t(t) \|_{\Gamma}^2 \, dt \right)$$

$$+ \mathcal{O}\left(\left[\int_{t_0}^{\infty} e^{-2\beta t} \| D^* w_t(t) \|_{\Gamma}^2 \, dt \right]^{1/2} \right) \cdot \mathcal{O}\left(\left[\int_{t_0}^{\infty} e^{-2\beta t} \| w(t) \|_{\Omega}^2 \, dt \right]^{1/2} \right) \tag{3.31}$$

where $\mathcal{O}$ denotes upper bound with a multiplicative constant independent *of β and t_0, and the right-hand side of* (3.31) *is finite by Corollary* 2.2 *and the contraction property of $E(t)$ in* (3.1).

Proof of Proposition 3.1. We prove initially equality (3.29) for $[w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A})$ and then extend to all of Z, using Remark 3.1 and Proposition 3.2. Adapting the multiplier technique used in [LLT1], we multiply both sides of the equation in (3.23) by $e^{-2\beta t} h \cdot \nabla p$ and integrate over $(t_0, \infty) \times \Omega$

$$\int_{\Omega} \int_{t_0}^{\infty} e^{-2\beta t} p_{tt} (h \cdot \nabla p) \, dt \, d\Omega$$

$$\equiv \int_{t_0}^{\infty} \int_{\Omega} e^{-2\beta t} \Delta p (h \cdot \nabla p) \, d\Omega \, dt + \int_{t_0}^{\infty} \int_{\Omega} e^{-2\beta t} F (h \cdot \nabla p) \, d\Omega \, dt. \tag{3.32}$$

As to the *left-hand side* (L.H.S.) of (3.32), we integrate by parts in t, use $e^{-2\beta T}(p_t(T), h \cdot \nabla p(T))_{\Omega} \to 0$ as $T \to \infty$ by (3.28), and $p_t \, (\partial/\partial t)(\partial p/\partial x_k) = \frac{1}{2}(\partial/\partial x_k)(p_t^2)$, thus obtaining:

left-hand side (L.H.S.) of (3.32) =

$$= -e^{-2\beta t_0}(p_t(t_0), h \cdot \nabla p(t_0))_{\Omega} + 2\beta \int_{t_0}^{\infty} \int_{\Omega} e^{-2\beta t} p_t h \cdot \nabla p \, d\Omega \, dt$$

$$- \frac{1}{2} \int_{t_0}^{\infty} \int_{\Omega} e^{-2\beta t} h \cdot \nabla (p_t^2) \, d\Omega \, dt. \tag{3.33}$$

From the identity

$$\operatorname{div}(\psi h) = h \cdot \nabla\psi + \psi \operatorname{div} h$$

valid for any, say, $H^1(\Omega)$-scalar function ψ, the divergence theorem gives

$$\int_\Omega h \cdot \nabla\psi \, d\Omega = \int_\Gamma \psi h \cdot \nu \, d\Gamma - \int_\Omega \psi \operatorname{div} h \, d\Omega \tag{3.34}$$

with ν outward unit normal. Application of (3.34) with $\psi = p_t^2$ in the last integral of (3.33) along with $p_t|_\Gamma \equiv 0$ (a consequence of (3.23c)) transforms (3.33) into

$$\begin{aligned}\text{L.H.S. of (3.32)} = {} & -e^{-2\beta t_0}(p_t(t_0), h \cdot \nabla p(t_0))_\Omega \\ & + 2\beta \int_{t_0}^\infty e^{-2\beta t} \int_\Omega p_t h \cdot \nabla p \, d\Omega \, dt + \tfrac{1}{2} \int_{t_0}^\infty e^{-2\beta t} \int_\Omega p_t^2 \operatorname{div} h \, d\Omega\end{aligned} \tag{3.35}$$

the $(\ ,\ )_\Omega$—inner product being well defined, see statement following (3.24). As to the *right-hand side* (R.H.S.) of (3.32), we apply Green's first theorem in its first term

$$\begin{aligned}&\int_{t_0}^\infty e^{-2\beta t} \int_\Omega \Delta p (h \cdot \nabla p) \, d\Omega \, dt \\ &\quad = \int_{t_0}^\infty e^{-2\beta t} \left[\int_\Gamma \frac{\partial p}{\partial \nu} (h \cdot \nabla p) \, d\Gamma - \int_\Omega \nabla p \cdot \nabla(h \cdot \nabla p) \, d\Omega \right] dt\end{aligned} \tag{3.36}$$

Direct computations show the following identity for any $H^2(\Omega)$—function Φ:

$$\begin{aligned}\nabla\Phi \cdot \nabla(h \cdot \nabla\Phi) &= \sum_{k=1}^n (\nabla\Phi \cdot \nabla h_k) \frac{\partial\Phi}{\partial x_k} + \frac{1}{2} h \cdot \nabla(|\nabla\Phi|^2) \\ \text{(by (3.30))} &= (H\nabla\Phi) \cdot \nabla\Phi + \tfrac{1}{2} h \cdot \nabla(|\nabla\Phi|^2).\end{aligned} \tag{3.37}$$

We next apply identity (3.37) with $\Phi = p$ to the last integral in (3.36)

$$\begin{aligned}\int_{t_0}^\infty e^{-2\beta t} \int_\Omega \Delta p (h \cdot \nabla p) \, d\Omega \, dt = {} & \int_{t_0}^\infty e^{-2\beta t} \int_\Gamma \frac{\partial p}{\partial \nu} (h \cdot \nabla p) \, d\Gamma \, dt \\ & - \int_{t_0}^\infty e^{-2\beta t} \int_\Omega (H\nabla p) \cdot \nabla p \, d\Omega \, dt \\ & - \tfrac{1}{2} \int_{t_0}^\infty e^{-2\beta t} \int_\Omega h \cdot \nabla(|\nabla p|^2) \, d\Omega \, dt.\end{aligned}$$

Finally, identity (3.34) with $\psi = |\nabla p|^2$ applied to the last integral above, along with $|\partial p/\partial v| \equiv |\nabla p|$ on Γ, (a consequence of the homogeneous B.C. (3.23c)), yields

R.H.S. of (3.32)

$$= \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} \frac{\partial p}{\partial v} (h \cdot \nabla p)\, d\Gamma\, dt - \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} (H \nabla p) \cdot \nabla p\, d\Omega\, dt$$

$$- \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} \left(\frac{\partial p}{\partial v} \right)^2 h \cdot v\, d\Gamma$$

$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} |\nabla p|^2 \operatorname{div} h\, d\Omega\, dt$$

$$+ \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} F(h \cdot \nabla p)\, d\Omega\, dt$$

Combining (3.35) and (3.38), we obtain

$$\int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} \left[\frac{\partial p}{\partial v} (h \cdot \nabla p) - \frac{1}{2} \left(\frac{\partial p}{\partial v} \right)^2 h \cdot v \right] d\Gamma\, dt$$

$$= \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} (H \nabla p) \cdot \nabla p\, d\Omega\, dt - \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} |\nabla p|^2 \operatorname{div} h\, d\Omega\, dt$$

$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} p_t^2 \operatorname{div} h\, d\Omega\, dt + 2\beta \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} p_t (h \cdot \nabla p)\, d\Omega\, dt$$

$$- e^{-2\beta t_0} (p_t(t_0), h \cdot \nabla p(t_0))_{\Omega} - \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} F(h \cdot \nabla p)\, d\Omega\, dt \tag{3.39}$$

But, from (3.24), integrating by parts in t

$$- \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} F(h \cdot \nabla p)\, d\Omega\, dt$$

$$= \int_{\Omega} \int_{t_0}^{\infty} DD^* w_{tt}(t) [e^{-2\beta t} h \cdot \nabla p(t)]\, dt\, d\Omega$$

$$= -e^{-2\beta t_0} (DD^* w_t(t_0), h \cdot \nabla p(t_0))_{\Omega}$$

$$+ 2\beta \int_{t_0}^{\infty} e^{-2\beta t} (DD^* w_t(t), h \cdot \nabla p(t))_{\Omega}\, dt$$

$$- \int_{t_0}^{\infty} e^{-2\beta t} (DD^* w_t(t), h \cdot \nabla p_t(t))_{\Omega}\, dt \tag{3.40}$$

since $e^{-2\beta T}(DD^*w_t(T), h\cdot\nabla p(T))_\Omega \to 0$ as $T\to\infty$ by (3.28). Now, the desired conclusion (3.29) follows from summing up (3.39), (3.40), after using (3.21): $p_t(t)+DD^*w_t(t)=-w(t)$ twice, once for $t=t_0$ in combining the two $(\ ,\ )_\Omega$-terms, and once in combining the two integrals premultiplied by 2β. Proposition 3.1 is proved. ∎

Proof of Proposition 3.2. Recalling (3.21) for p_t, we write

$$\int_{t_0}^\infty e^{-2\beta t}(DD^*w_t(t), h\cdot\nabla p_t(t))_\Omega\,dt = I_1+I_2 \tag{3.41}$$

$$I_1 = -\int_{t_0}^\infty e^{-2\beta t}(DD_t^*w(t), h\cdot\nabla(DDw_t^*(t)))_\Omega\,dt \tag{3.42}$$

$$I_2 = -\int_{t_0}^\infty e^{-2\beta t}(DD^*w_t(t), h\cdot\nabla w(t))_\Omega\,dt \tag{3.43}$$

To estimate I_1 and I_2, we need the following extension of (3.34)

$$\int_\Omega gh\cdot\nabla\psi\,d\Omega = \int_\Gamma g\psi h\cdot\nu\,d\Gamma - \int_\Omega \psi h\cdot\nabla g\,d\Omega - \int_\Omega g\psi\operatorname{div} h\,d\Omega \tag{3.44a}$$

for scalar, say $H^1(\Omega)$-functions g and ψ, easily obtainable from (3.34), which for $g=\psi$ specializes to[2]

$$\int_\Omega \psi h\cdot\nabla\psi\,d\Omega = \tfrac{1}{2}\int_\Gamma \psi^2 h\cdot\nu\,d\Gamma - \tfrac{1}{2}\int_\Omega \psi^2\operatorname{div} h\,d\Omega. \tag{3.44b}$$

To estimate I_1, we use (3.44b) with $\psi = DD^*w_t$, whereby $\psi|_\Gamma = D^*w_t$, so that

$$I_1 = \mathcal{O}\left(\int_{t_0}^\infty e^{-2\beta t}\,\|D^*w_t(t)\|_\Gamma^2\,dt\right) \tag{3.45}$$

by virtue also of D being continuous $L_2(\Gamma)\to L_2(\Omega)$, (1.6), and Corollary 2.2. To estimate I_2, we first rewrite (3.44a) with $g=DD^*w_t$, $g|_\Gamma = D^*w_t$ and $\psi = w$, so that $\psi|_\Gamma = w|_\Gamma = -D^*w_t$ by (1.18c), thus obtaining

$$\begin{aligned}(DD^*w_t(t), h\cdot\nabla w(t))_\Omega = &-\int_\Gamma (D^*w_t(t))\,D^*w_t(t)\,h\cdot\nu\,d\Gamma\\ &-(w(t), DD^*w_t(t)\operatorname{div} h)_\Omega\\ &-(w(t), h\cdot\nabla(DD^*w_t(t))_\Omega\end{aligned} \tag{3.46}$$

[2] (3.44b) follows more directly by applying the divergence theorem to the vector field $G=\frac{1}{2}\psi^2 h$, so that $\operatorname{div} G = \psi(h\cdot\nabla\psi)+\frac{1}{2}\psi^2\operatorname{div} h$.

where the first two terms on the right-hand side of (3.46) are well-defined at least a.e. in t, by Corollary 2.2. Moreover, via (1.6), they can be directly estimated as

$$\int_{\Gamma} (D^*w_t(t))^2 h\cdot \nu \, d\Gamma + (w(t), DD^*w_t(t) \operatorname{div} h)_{\Omega}$$
$$= \mathcal{O}(\|D^*w_t(t)\|_{\Gamma}^2) + \mathcal{O}(\|w(t)\|_{\Omega} \|D^*w_t(t)\|_{\Gamma}) \qquad \text{a.e. in } t. \quad (3.47)$$

That the third term on the right of (3.46) is also well-defined a.e. in t is the result of the following:

LEMMA 3.3. *Under the assumptions of Proposition 3.2 on h, so that $h(\sigma) = k(\sigma)\nu$ on Γ, $k(\sigma) \in C(\Gamma)$, we have*

$$(w(t), h\cdot\nabla(DD^*w_t(t))_{\Omega} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \nu}(DD^*w(t)), kD^*w(t))\right)_{\Gamma}$$
$$+ \mathcal{O}(\|w(t)\|_{\Omega} \|D^*w_t(t)\|_{\Gamma}) \qquad \text{a.e. in } t. \qquad (3.48)$$

Proof. See Appendix A. ∎

Continuing with the proof of Proposition 3.2, we return to (3.46), where we use (3.47)–(3.48), thus obtaining

$$(DD^*w_t(t), h\cdot\nabla w(t))_{\Omega} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \nu}(DD^*w(t)), kD^*w(t)\right)_{\Gamma}$$
$$+ \mathcal{O}(\|w(t)\|_{\Omega} \|D^*w_t(t)\|_{\Gamma})$$
$$+ \mathcal{O}(\|D^*w_t(t)\|_{\Gamma}^2), \quad \text{a.e. in } t. \qquad (3.49)$$

By means of (3.49), we can finally estimate I_2 in (3.43).

$$I_2 = \frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \nu}(DD^*w(t)), kD^*w(t)\right)_{\Gamma} dt$$
$$+ \int_{t_0}^{\infty} \mathcal{O}\{e^{-\beta t}\|w(t)\|_{\Omega}\, e^{-\beta t}\|D^*w_t(t)\|\}\, dt$$
$$+ \int_{t_0}^{\infty} e^{-2\beta t}\mathcal{O}(\|D^*w_t(t)\|_{\Gamma}^2)\, dt = -\frac{1}{2}e^{-2\beta t_0}\left(\frac{\partial}{\partial \nu}(DD^*w(t_0)), kD^*w(t_0)\right)_{\Gamma}$$
$$+ \frac{1}{2}\cdot 2\beta\int_{t_0}^{\infty} e^{-2\beta t}\left(\frac{\partial}{\partial \nu}(DD^*w(t)), kD^*w(t)\right)_{\Gamma} dt$$
$$+ \mathcal{O}\left\{\left[\int_{t_0}^{\infty} e^{-2\beta t}\|w(t)\|_{\Omega}^2\, dt\right]^{1/2}\right\} \mathcal{O}\left\{\left[\int_{t_0}^{\infty} e^{-2\beta t}\|D^*w_t(t)\|_{\Gamma}^2\, dt\right]^{1/2}\right\}$$
$$+ \mathcal{O}\left\{\int_{t_0}^{\infty} e^{-2\beta t}\|D^*w_t(t)\|_{\Gamma}^2\, dt\right\} \qquad (3.50)$$

where, in the last step, we have used that, for $\beta > 0$

$$e^{-2\beta T}\left(\frac{\partial}{\partial \nu}(DD^*w(T), kD^*w(T)\right)_\Gamma \to 0 \qquad \text{as} \quad T \to \infty. \tag{3.51}$$

This is so, since $w(T) \in L_2(\Omega)$, hence $D^*w(T) \in H^{1/2}(\Omega)$ by (1.9), finally $DD^*w(T) \in H^1(\Omega)$ by (1.6) and solves the Laplace equation. Therefore, $(\partial/\partial\nu)\, DD^*w(T) \in H^{-1/2}(\Gamma)$ [Kellogg theorem 3.8.1 p. 71 and ff] and

$$\left|\left(\frac{\partial}{\partial \nu} DD^*w(T), kD^*w(T)\right)_\Gamma\right| \leqslant c \left\|\frac{\partial}{\partial \nu} DD^*w(T)\right\|_{H^{-1/2}(\Gamma)} \|D^*w(T)\|_{H^{1/2}(\Gamma)}$$

$$(c = \max_{\sigma \in \Gamma} |k(\sigma)|) \qquad \leqslant C\, \|w(T)\|^2_\Omega \leqslant CE(T) \leqslant CE(t_0) \tag{3.52}$$

for all $T \geqslant t_0$, by contraction of $E(\ )$, ((3.1)), and (3.51) follows from (3.52). Using again (3.52) for the first two terms in the expression of I_2 in (3.50), we then obtain

$$\begin{aligned} I_2 = \mathcal{O}(E(t_0)) + \beta\mathcal{O}\left\{\int_{t_0}^\infty e^{-2\beta t}\, \|w(t)\|^2_\Omega\, dt\right\} + \mathcal{O}\left\{\int_{t_0}^\infty e^{-2\beta t}\, \|D^*w_t(t)\|^2_\Gamma\, dt\right\} \\ + \mathcal{O}\left\{\left[\int_{t_0}^\infty e^{-2\beta t}\, \|w(t)\|^2_\Omega\, dt\right]^{1/2}\right\} \mathcal{O}\left\{\left[\int_{t_0}^\infty e^{-2\beta t}\, \|D^*w_t(t)\|^2_\Gamma\, dt\right]^{1/2}\right\} \end{aligned} \tag{3.53}$$

The desired estimate (3.31) of Proposition 3.2 now follows from (3.41), (3.45), and (3.53).

3.3. *Use of Assumption* (H.2) = (1.21) *and Final Estimate in Terms of p*

Having proved Propositions 3.1 and 3.2, we continue with the proof of the main theorem. We now use the estimate (3.31) of Proposition 3.2 for the last term in identity (3.29) of Proposition 3.1 and obtain

$$\int_{t_0}^\infty e^{-2\beta t} \int_\Omega (H\nabla p)\cdot\nabla p\, d\Omega\, dt - \frac{1}{2}\int_{t_0}^\infty e^{-2\beta t}\int_\Omega |\nabla p|^2 \operatorname{div} h\, d\Omega\, dt$$

$$+ \frac{1}{2}\int_{t_0}^\infty e^{-2\beta t}\int_\Omega p_t^2 \operatorname{div} h\, d\Omega\, dt$$

$$= \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} \left[\frac{\partial p}{\partial \nu} (h \cdot \nabla p) - \frac{1}{2} \left(\frac{\partial p}{\partial \nu} \right)^2 h \cdot \nu \right] d\Gamma\, dt$$

$$+ 2\beta \int_{t_0}^{\infty} e^{-2\beta t} (w, h \cdot \nabla p)_{\Omega}\, dt$$

$$- e^{-2\beta t_0} (w(t_0), h \cdot A^{-1/2} w_t(t_0))_{\Omega} + \mathcal{O}(E(t_0))$$

$$+ \beta \mathcal{O} \left(\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_{\Omega}^2\, dt \right) + \mathcal{O} \left(\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2\, dt \right)$$

$$+ \mathcal{O} \left(\left[\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2\, dt \right]^{1/2} \right) \mathcal{O} \left(\left[\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_{\Omega}^2\, dt \right]^{1/2} \right). \tag{3.54}$$

But, recalling the assumption $h|_{\Gamma} = k(\sigma)\nu$ on Γ and the relations $(\partial p/\partial \nu) = -D^* w_t$ ((3.19)), and $|\partial p/\partial \nu| = |\nabla p|$ on Γ (by (3.23c)), we have

$$\int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} \left[\frac{\partial p}{\partial \nu} (h \cdot \nabla p) - \frac{1}{2} \left(\frac{\partial p}{\partial \nu} \right)^2 h \cdot \nu \right] d\Gamma\, dt$$

$$= \mathcal{O} \left(\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2\, dt \right). \tag{3.55}$$

Moreover, by Schwarz inequality and (3.26)

$$2\beta \int_{t_0}^{\infty} e^{-2\beta t} (w, h \cdot \nabla p)_{\Omega}\, dt$$

$$= \beta \mathcal{O} \left\{ \left[\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_{\Omega}^2\, dt \right]^{1/2} \right\} \mathcal{O} \left\{ \left[\int_{t_0}^{\infty} e^{-2\beta t} \| |\nabla p(t)| \|_{\Omega}^2 \right]^{1/2} \right\}$$

$$= \beta \mathcal{O} \left\{ \int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt \right\}. \tag{3.56}$$

Similarly

$$e^{-2\beta t_0} (w(t_0), h \cdot A^{-1/2} w_t(t_0))_{\Omega} = \mathcal{O}(E(t_0)). \tag{3.57}$$

Using (3.55)–(3.57) in (3.54) yields the simplified expression

$$\int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} (H \nabla p) \cdot \nabla p\, d\Omega\, dt - \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} |\nabla p|^2 \operatorname{div} h\, d\Omega\, dt$$

$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} p_t^2 \operatorname{div} h\, d\Omega\, dt$$

$$= \mathcal{O}(E(t_0)) + \beta \mathcal{O} \left\{ \int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt \right\} + \mathcal{O} \left\{ \int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2 \, dt \right\}$$
$$+ \mathcal{O} \left\{ \left[\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2 \, dt \right]^{1/2} \right\} \mathcal{O} \left\{ \left[\int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt \right]^{1/2} \right\}. \tag{3.58}$$

Using now assumption (H2) = (1.21) on H on the left-hand side of (3.58) and $2a^{1/2}b^{1/2} \leqslant (1/\varepsilon_2)a + \varepsilon_2 b$ on the last term of the right-hand side, we finally arrive at the following result, which we formalize as

PROPOSITION 3.4. *Under the vector field assumption on Ω, the following estimate holds for $\beta > 0$*

$$\rho \int_{t_0}^{\infty} e^{-2\beta t} \| |\nabla p(t)| \|_{\Omega}^2 \, dt - (\beta + \varepsilon_2)\, \mathcal{O} \left\{ \int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt \right\}$$
$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} (p_t^2(t) - |\nabla p(t)|^2) \operatorname{div} h \, d\Omega \, dt$$
$$\leqslant \mathcal{O}(E(t_0)) + \left(1 + \frac{1}{\varepsilon_2}\right) \mathcal{O} \left\{ \int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2 \, dt \right\} \tag{3.59}$$

where $\varepsilon_2 > 0$ is arbitrary.

3.4. *Return Form p to w in Estimate* (3.59)

We now express the left hand side of (3.59) in terms of w. We begin with a Lemma on the third integral term of (3.59).

LEMMA 3.5. *Under the assumptions of Proposition* 3.4 *we have for $\beta > 0$.*

$$\frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} (p_t^2(t) - |\nabla p(t)|^2) \operatorname{div} h \, d\Omega \, dt$$
$$= G_2(w) + (1 + |r|) \left\{ (\varepsilon_1 + \varepsilon_3)\, \mathcal{O} \left(\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_{\Omega}^2 \, dt \right) \right.$$
$$\left. + \left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3}\right) \mathcal{O} \left(\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2 \, dt \right) \right\}$$
$$+ \varepsilon_4\, \mathcal{O} \left(\int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|_{\Omega}^2 \, dt \right) + \frac{1}{\varepsilon_4} \mathcal{O} \left(\int_{t_0}^{\infty} e^{-2\beta t} \|p(t)\|_{\Omega}^2 \, dt \right) \tag{3.60}$$

where $G_2(w)$ is defined by (3.15), *ε_1 is as in* (3.17), *and ε_1, ε_3, and ε_4 are arbitrary positive constants.*

Proof. We begin with the term involving $p_t^2 = w^2 + (DD^*w_t)^2 + 2w(DD^*w_t)$, (from (3.21))

$$\frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Omega} p_t^2(t) \operatorname{div} h \, d\Omega \, dt$$

$$=\frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t}(w(t), w(t) \operatorname{div} h)_{\Omega} \, dt$$

$$+\frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t}(DD^*w_t(t), DD^*w_t(t) \operatorname{div} h)_{\Omega} \, dt$$

$$+\int_{t_0}^{\infty} e^{-2\beta t}(DD^*w_t(t), w(t) \operatorname{div} h)_{\Omega} \, dt \tag{3.61}$$

Also writing $\nabla(p \operatorname{div} h) = \operatorname{div} h \nabla p + p\nabla(\operatorname{div} h)$, we have

$$\int_{\Omega} |\nabla p|^2 \operatorname{div} h \, d\Omega = \int_{\Omega} \operatorname{div} h \nabla p \cdot \nabla p \, d\Omega$$

$$= \int_{\Omega} \nabla p \cdot \nabla(p \operatorname{div} h) \, d\Omega - \int_{\Omega} \nabla p \cdot \nabla(\operatorname{div} h) \, p \, d\Omega$$

(Green's first theorem, with $p = 0$ on Γ as in (3.23c)).

$$= -\int_{\Omega} \Delta p(p \operatorname{div} h) \, d\Omega - \int_{\Omega} [\nabla p \cdot \nabla(\operatorname{div} h)] \, p \, d\Omega$$

(using $w_t = AA^{-1}w_t = -\Delta(A^{-1}w_t) = -\Delta p$, by (3.20), as well as (3.26))

$$= (w_t, A^{-1}w_t \operatorname{div} h)_{\Omega} + \mathcal{O}(\|A^{-1/2}w_t\|_{\Omega} \|p\|_{\Omega}). \tag{3.62}$$

Combining (3.61) with (3.62)

$$\frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Omega} (p_t^2(t) - |\nabla p(t)|^2) \operatorname{div} h \, d\Omega \, dt$$

$$=\frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t}[(w(t), w(t) \operatorname{div} h)_{\Omega} - (w_t(t), A^{-1}w_t(t) \operatorname{div} h)_{\Omega}] \, dt$$

$$+\frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t}(DD^*w_t(t), DD^*w_t(t) \operatorname{div} h)_{\Omega}$$

$$+\int_{t_0}^{\infty} e^{-2\beta t}(DD^*w_t(t), w(t) \operatorname{div} h)_{\Omega} \, dt$$

$$+\int_{t_0}^{\infty} e^{-2\beta t}\mathcal{O}(\|A^{-1/2}w_t(t)\|_{\Omega} \, \|p(t)\|_{\Omega}) \, dt \tag{3.63}$$

from which via Schwartz inequality we obtain with $D: L_2(\Gamma) \to L_2(\Omega)$

$$\frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} (p_t^2(t) - |\nabla p(t)|^2) \operatorname{div} h \, d\Omega \, dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t} [w(t) \operatorname{div} h, w(t))_{\Omega} - (A^{-1} w_t(t) \operatorname{div} h, w_t(t))_{\Omega}] \, dt$$

$$+ \mathcal{O}\left\{\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2 \, dt\right\}$$

$$+ \mathcal{O}\left\{\int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|_{\Omega} \|p(t)\|_{\Omega} \, dt\right\}$$

$$+ \mathcal{O}\left\{\left[\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2 \, dt\right]^{1/2}\right\}$$

$$\times \mathcal{O}\left\{\left[\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_{\Omega}^2 \, dt\right]^{1/2}\right\}. \tag{3.64}$$

Recalling (3.17) and using $2a^{1/2}b^{1/2} \leqslant (1/\varepsilon)a + \varepsilon b$ in the last two additive terms of (3.64), we obtain the desired relation (3.60). ∎

We now use (3.60) of Lemma 3.5 into (3.59) of Proposition 3.4. Writing $\| |\nabla p(t)| \|_{\Omega}^2 = \|A^{-1/2} w_t(t)\|_{\Omega}^2$, see (3.26) and rearranging the terms, we finally arrive at

PROPOSITION 3.6. *Under the assumptions of Lemma 3.5, we have for* $\beta > 0$

$$G_2(w) \leqslant -\rho \int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|_{\Omega}^2 \, dt + (\beta + \varepsilon_2)\mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} E(t) \, dt\right)$$

$$+ (1 + |r|)\left\{(\varepsilon_1 + \varepsilon_3)\mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_{\Omega}^2 \, dt\right)\right.$$

$$\left. + \left(2 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right) \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_{\Gamma}^2 \, dt\right)\right\}$$

$$+ \varepsilon_4 \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|_{\Omega}^2 \, dt\right)$$

$$+ \frac{1}{\varepsilon_4} \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|p(t)\|_{\Omega}^2 \, dt\right) + \mathcal{O}(E(t_0)) \tag{3.65}$$

with ε_i *arbitrary positive numbers.*

Proposition 3.6 represents a first major step in the effort to achieve the decomposition of $G_2(w)$ in the desirable form, which was explained in points (i), (ii), (iii) of the *Orientation*, given below (3.16). The second and conclusive major step in this direction is given by the following proposition on the term before the last in (3.65).

PROPOSITION 3.7. *For any* $\varepsilon > 0$, *there exists a number* $C_\varepsilon > 0$ *such that for every* $\beta > 0$ *and* $t_0 > 0$

$$\int_{t_0}^{\infty} e^{-2\beta t} \|p(t)\|_\Omega^2 \, dt \leqslant \varepsilon \int_{t_0}^{\infty} e^{-2\beta t} \|p_t(t)\|_\Omega^2 \, dt + (t_0 + 1) C_\varepsilon E(0). \quad (3.66)$$

Proof. See Section 3.5 below. ∎

Once in possession of estimates (3.65) and (3.66), we can finally achieve our goal of proving inequality (3.16). By (3.21)

$$\int_{t_0}^{\infty} e^{-2\beta t} \|p_t(t)\|_\Omega^2 \, dt \leqslant 2 \int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_\Omega^2 \, dt + 2K \int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2 \, dt \quad (3.67)$$

($K = \|D^*\|^2_{\mathscr{L}(L_2(\Omega), L_2(\Gamma))}$). Using (3.65)–(3.67) and recalling (3.13), we obtain

$$\begin{aligned}
&G_1(w) + G_2(w) - 2qt_0 \int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2 \, dt \\
&\quad \leqslant q \int_{t_0}^{\infty} e^{-2\beta t} E(t) \, dt + (\beta + \varepsilon_2)\, \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} E(t) \, dt \right) \\
&\quad\quad + \frac{r}{2} \int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|_\Omega^2 \, dt \\
&\quad\quad - \rho C_1^2 \int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|_\Omega^2 \, dt \\
&\quad\quad + \varepsilon_4 \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|_\Omega^2 \, dt \right) \\
&\quad\quad - \frac{r}{2} \int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_\Omega^2 \, dt + (1 + |r|)(\varepsilon_1 + \varepsilon_3) \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_\Omega^2 \, dt \right) \\
&\quad\quad + \frac{1}{\varepsilon_4} \cdot 2\varepsilon \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|_\Omega^2 \, dt \right) - 2qt_0 \int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2 \, dt \\
&\quad\quad + \left[(1 + |r|) \left(2 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} \right) + \frac{2K\varepsilon}{\varepsilon_4} \right] \mathcal{O} \\
&\quad\quad \times \left(\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|_\Gamma^2 \, dt \right) + \mathcal{O}(E(t_0) + C_\varepsilon \int_0^{t_0} E(t) \, dt.
\end{aligned} \quad (3.68)$$

Then, our goal of establishing inequality (3.16) is a fortiori accomplished, *provided we prove that* the right-hand side (R.H.S.) of (3.68) satisfies

$$\text{R.H.S. of (3.68)} \leqslant -C^2_{q,r,t_0} \int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt + C'^2_{q,r,t_0} E(t_0) \qquad (3.39)$$

for a suitable selection of the parameters q, $t_0 > 0$, and r, and for suitable constants C^2, C'^2 depending on these parameters. We recall, at this point, that the symbol $\mathcal{O}$ in (3.68) denotes upper bound by a constant, which is *independent of* the parameter $\beta > 0$ as well as of the parameters ε, $\varepsilon_i > 0$ and t_0, $i = 1,..., 4$. Moreover, as explained in the paragraph between (3.11a) and (3.11b), we eventually let $\beta \downarrow 0$. Then, to achieve (3.69), we proceed through the following steps by analyzing the R.H.S. of (3.68):

(i) given $\rho > 0$ (ρ as in (1.21)), we choose ε_4 so small as to have negative the coefficient of $\int_{t_0}^{\infty} e^{-2\beta t} \|A^{-1/2} w_t(t)\|^2_\Omega\, dt$ in the R.H.S. of (3.68), i.e., $R_1 \equiv [r/2 - \rho + \varepsilon_4 \mathcal{O}] < 0$, hence $r/2 < \rho$;

(ii) next, we choose ε_1, ε_3, ε so small as to have negative the coefficient of $\int_{t_0}^{\infty} e^{-2\beta t} \|w(t)\|^2_\Omega\, dt$ in the R.H.S. of (3.68), i.e.,

$$R_2 \equiv \left[-\frac{r}{2} + (1 + |r|)(\varepsilon_1 + \varepsilon_3) + \frac{1}{\varepsilon_4} 2\varepsilon \mathcal{O} \right] < 0$$

whereby $r > 0$.

(iii) setting $R = \max[R_1, R_2] < 0$, we then select β and ε_2 so small that

$$q + (\beta + \varepsilon_2)\mathcal{O} + R < 0$$

for suitable $0 < q < |R|$;

(iv) having fixed ε_i, ε, $i = 1,..., 4$ and selected r and q as above, we then choose $t_0 > 0$ so large as to have negative the coefficient of $\int_{t_0}^{\infty} e^{-2\beta t} \|D^* w_t(t)\|^2_\Gamma\, dt$ in the R.H.S. of (3.68), i.e.,

$$-2qt_0 + \left[(1 + |r|) \left(2 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} \right) + \frac{2K\varepsilon}{\varepsilon_4} \right] \mathcal{O} < 0$$

in agreement with the strategy outlined in the *Orientation* below (3.16). This way, (3.69) is proved, and so is (3.16). The proof of the main theorem is thus complete, as soon as we show Proposition 3.7.

3.5. *Proof of Proposition* 3.7

As in [L1], we shall employ Laplace transform techniques in the variable $\lambda = \beta + i\alpha$, $\alpha \in R$, and obtain the needed estimates for small $|\alpha|$ and large $|\alpha|$ separately. However, unlike [L1], we prefer to use our operator model for (3.72) below rather than the arguments in [L1]. As in [L1], we introduce a new variable

$$u(t, x) = \Phi(t)\, p(t, x); \qquad \Phi \in C^\infty(R); \Phi(0) = \Phi'(0) = 0; \Phi(t) \equiv 1,\ t \geq t_0 \tag{3.70a}$$

$$u_t = p_t \Phi + p\Phi' \tag{3.70b}$$

so that

$$\int_{t_0}^\infty e^{-2\beta t} \|p(t)\|_\Omega^2\, dt = \int_{t_0}^\infty e^{-2\beta t} \|u(t)\|_\Omega^2\, dt \leq \int_0^\infty e^{-2\beta t} \|u(t)\|_\Omega^2\, dt. \tag{3.71}$$

Then, in the new variable u problem (3.23) becomes

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + B \qquad \text{in } (0, \infty) \times \Omega \tag{3.72a}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in } \Omega \tag{3.72b}$$

$$u = 0 \qquad \text{in } (0, \infty) \times \Gamma \tag{3.72c}$$

where by $p_t = A^{-1} w_{tt}$ [(3.21)] and (3.70b)

$$B = -DD^* w_{tt} \Phi + p\Phi'' + 2p_t \Phi' \tag{3.73a}$$

$$DD^* w_{tt} \Phi = DD^* A p_t \Phi = DD^* A[u_t - p\Phi']. \tag{3.73b}$$

The solution to problem (3.72) is then by (3.73)

$$\begin{aligned} u(t) &= \int_0^t S(t-\tau)\, B(\tau)\, d\tau \\ &= -\int_0^t S(t-\tau)\, DD^* A u_t(\tau)\, d\tau + 2\int_0^t S(t-\tau)\, p_t(\tau)\, \Phi'(\tau)\, d\tau \\ &\quad + \int_0^t S(t-\tau)[DD^* A p(\tau)\, \Phi'(\tau) + p(\tau)\, \Phi''(\tau)]\, d\tau, \end{aligned} \tag{3.74}$$

$S(t)y = \int_0^t C(\tau) y\, d\tau$, $C(\ )$ the cosine operator on $L_2(\Omega)$ generated by $-A$. Taking the Laplace transform ($\mathscr{L}$) of (3.74) with $\hat{u}_t(\lambda) = \lambda \hat{u}(\lambda)$ by (3.72b) we obtain via $\mathscr{L}[S(t)] = R(\lambda^2, -A)$:

$$\begin{aligned} \hat{u}(\lambda) = &-\lambda R(\lambda^2, -A)\, DD^* A \hat{u}(\lambda) + 2R(\lambda^2, -A) \mathscr{L}[p_t \Phi'](\lambda) \\ &+ R(\lambda^2, -A)\, \mathscr{L}[DD^* A p\Phi' + p\Phi''](\lambda) \end{aligned}$$

i.e.

$$[I+\lambda R(\lambda^2, -A)\, DD^*A]\, \hat{u}(\lambda)$$
$$= R(\lambda^2, -A)\{\mathscr{L}[DD^*Ap\Phi' + p\Phi'' + 2p_t\Phi'](\lambda)\} \tag{3.75}$$

valid at least for all $\lambda = \beta + i\alpha$, $\operatorname{Re}\lambda \geqslant 0$, with $\lambda^2 = \beta^2 - \alpha^2 + 2i\alpha\beta \neq \{-\mu_n, n = 1, 2,...\}$, $\mu_n > 0$ being the eigenvalues of the positive self adjoint operator A (as in Sect. 2); i.e., except when: $\beta = 0$ and $\alpha^2 = \mu_n$, i.e., $\lambda = \lambda_n = \pm i\sqrt{\mu_n}$, where $R(\lambda^2, -A)$ is *not* defined. However, noticing that

$$I + \lambda R(\lambda^2, -A)\, DD^*A = R(\lambda^2, -A)[I + \lambda DD^* + \lambda^2 A^{-1}]A \tag{3.76}$$

$\operatorname{Re}\lambda \geqslant 0$, $\lambda \neq \lambda_n$, we have from (3.75)–(3.76), using $(\lambda^2 + A)\, R(\lambda^2, -A) =$ identity on $L_2(\Omega)$, that:

$$\hat{u}(\lambda) = A^{-1}[I + \lambda DD^* + \lambda^2 A^{-1}]^{-1}\{\mathscr{L}[DD^*Ap\Phi' + p\Phi'' + 2p_t\Phi'](\lambda)\} \tag{3.77}$$

Now, because of the results in Section 2 (see operator (2.3) in Lemma 2.1), we know that the operator $V(\lambda) = [I + \lambda DD^* + \lambda^2 A^{-1}]^{-1}$ is well defined as a bounded operator on $L_2(\Omega)$ in the *closed* right half-plane $\operatorname{Re}\lambda \geqslant 0$, including the imaginary axis $\beta = 0$, and is holomorphic in $\operatorname{Re}\lambda > 0$ [K1, p. 365 bottom]. Moreover, for any λ in the *closed* rectangle, say $\mathscr{R}_{\alpha_0}: 0 \leqslant \operatorname{Re}\lambda \leqslant 1$, $|\operatorname{Im}\lambda| \leqslant \alpha_0$, with $\alpha_0 > 0$ arbitrary, we have

$$\|A^{-1}V(\lambda)\|_{\mathscr{L}(L_2(\Omega))} = \|A^{-1}[I + \lambda DD^* + \lambda^2 A^{-1}]^{-1}\|_{\mathscr{L}(L_2(\Omega))} \leqslant C_{\alpha_0}, \quad \lambda \in \mathscr{R}_{\alpha_0}. \tag{3.78}$$

Recalling $Ap = w_t$ and $p_t = -[w + DD^*w_t]$ from (3.20)–(3.21), we rewrite (3.77) as

$$\hat{u}(\lambda) = A^{-1}V(\lambda)\{\mathscr{L}[-\Phi' DD^*w_t + \Phi''(A^{-1}w_t) - 2\Phi' w](\lambda)\}. \tag{3.79}$$

By (3.78)–(3.79), we then obtain with $\lambda = \beta + i\alpha$, β fixed $0 < \beta \leqslant 1$, using Parseval equality)

$$\int_{|\alpha| \leqslant \alpha_0} \|\hat{u}(\beta + i\alpha)\|_\Omega^2\, d\alpha$$
$$\leqslant CC_{\alpha_0}\left\{\int_{|\alpha| \leqslant \alpha_0} \|\mathscr{L}[\Phi' DD^*w_t](\beta + i\alpha)\|_\Omega^2\, d\alpha\right.$$
$$+ \int_{|\alpha| \leqslant \alpha_0} \|\mathscr{L}[\Phi''(A^{-1}w_t)](\beta + i\alpha)\|_\Omega^2\, d\alpha$$
$$\left. + \int_{|\alpha| \leqslant \alpha_0} \|\mathscr{L}[\Phi' w](\beta + i\alpha)\|_\Omega^2\, d\alpha\right\}$$

$$\leqslant CC_{\alpha_0}\left\{\int_0^\infty e^{-2\beta t}\|\Phi'(t)\, DD^* w_t(t)\|_\Omega^2\, dt\right.$$

$$+\int_0^\infty e^{-2\beta t}\|\Phi''(t)(A^{-1}w_t)(t)\|_\Omega^2\, dt$$

$$\left.+\int_0^\infty e^{-2\beta t}\,\|\Phi'(t)\, w(t)\|_\Omega^2\, dt\right\} \tag{3.80}$$

(recalling from (3.70a) that $\Phi' \equiv \Phi'' \equiv 0$ for $t \geqslant t_0$, and invoking (1.6) and (3.1))

$$\leqslant CC_{\alpha_0}\left\{\int_0^{t_0}\|D^* w_t(t)\|_\Gamma^2\, dt + \int_0^{t_0} E(t)\, dt\right\}.$$

By using Corollary 2.2 on the first integral and contraction of $E(t)$ on the second, we obtain

$$\int_{|\alpha| \leqslant \alpha_0}\|\hat{u}(\beta + i\alpha)\|_\Omega^2\, d\alpha \leqslant CC_{\alpha_0}[E(0) + t_0 E(0)]$$

$$= C(t_0+1)\, C_{\alpha_0} E(0),\ \lambda = \beta + i\alpha \in \mathscr{R}_{\alpha_0}. \tag{3.81}$$

For $\lambda = \beta + i\alpha$, $0 < \beta \leqslant 1$, $|\alpha| > \alpha_0 > 0$, we proceed as in [L1]: $\hat{u}_t(\lambda) = \lambda\hat{u}(\lambda)$ by (3.72b) and so, since $1/|\lambda|^2 \leqslant (|\alpha|^2/\alpha_0^2)(1/|\lambda|^2) \leqslant 1/\alpha_0^2$

$$\int_{|\alpha| > \alpha_0}\|\hat{u}(\lambda)\|_\Omega^2\, d\alpha = \int_{|\alpha| > \alpha_0}\frac{1}{|\lambda|^2}\|\lambda\hat{u}(\lambda)\|_\Omega^2\, d\alpha$$

$$\leqslant \frac{1}{\alpha_0^2}\int_{|\alpha| > \alpha_0}\|\lambda\hat{u}(\lambda)\|_\Omega^2\, d\alpha \leqslant \frac{2\pi}{\alpha_0^2}\int_0^\infty e^{-2\beta t}\,\|u_t(t)\|_\Omega^2\, dt \tag{3.82}$$

by Parseval equality. Taking now $1/\alpha_0^2 \leqslant \varepsilon$, ε preassigned, and using (3.81)–(3.82) and Parseval equality, we get

$$\int_0^\infty e^{-2\beta t}\,\|u(t)\|_\Omega^2\, dt = \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{u}(\beta + i\alpha)\|_\Omega^2\, d\alpha$$

$$\leqslant \varepsilon\int_0^\infty e^{-2\beta t}\,\|u_t(t)\|_\Omega^2\, dt + (t_0+1)\, CC_{\alpha_0} E(0). \tag{3.83}$$

Since $u \equiv p$ for $t \geqslant t_0$,

$$\int_0^\infty e^{-2\beta t}\,\|p(t)\|_\Omega^2\, dt \leqslant \int_0^{t_0}\|p(t)\|_\Omega^2\, dt + \int_0^\infty e^{-2\beta t}\,\|u(t)\|_\Omega^2\, dt$$

$$\text{(by (3.20) and (3.83))} \leqslant \int_0^{t_0} \|(A^{-1}w_t)(t)\|_\Omega^2\,dt$$

$$+\varepsilon \int_0^\infty e^{-2\beta t}\,\|u_t(t)\|_\Omega^2\,dt + 2t_0 CC_\varepsilon E(0)$$

$$\leqslant \varepsilon \int_0^\infty e^{-2\beta t}\,\|u_t(t)\|_\Omega^2\,dt + (t_0 + 2t_0 CC_\varepsilon)\,E(0) \tag{3.84}$$

by using $\|(A^{-1}w_t)(t)\|_\Omega^2 \leqslant E(t) \leqslant E(0)$ for $t_0 \geqslant t \geqslant 0$. By $u_t = p_t\Phi + p\Phi'$, $\Phi \equiv 1$, for $t > t_0$, ((3.70b)),

$$\int_0^\infty e^{-2\beta t}\,\|u_t(t)\|_\Omega^2\,dt = C_\Phi\left[\int_0^{t_0} \|p(t)\|_\Omega^2\,dt + \int_{t_0}^\infty e^{-2\beta t}\,\|p_t(t)\|_\Omega^2\,dt\right]. \tag{3.85}$$

But, by (3.20), (3.21), (1.6) and Corollary 2.2

$$\|p(t)\|_\Omega^2 + \|p_t(t)\|_\Omega^2 \leqslant C\{\|(A^{-1}w_t)(t)\|_\Omega^2 + \|w(t)\|_\Omega^2 + \|D^*w(t)\|_\Gamma^2 \leqslant CE(0)$$

so that (3.85) becomes

$$\int_0^\infty e^{-2\beta t}\,\|u_t(t)\|_\Omega^2\,dt \leqslant Ct_0E(0) + \int_{t_0}^\infty e^{-2\beta t}\,\|p_t(t)\|_\Omega^2\,dt. \tag{3.86}$$

Finally (3.86), inserted into (3.84), produces (3.66) as desired. Proposition 3.7 is proved. ∎

4. Wave Equation with Dissipative Feedback in the Neumann Boundary Conditions: a Sketch

For sake of completeness, we provide in this section a sketchy treatment—in the spirit of Sections 2 and 3—of the corresponding energy decay problem of the wave equation on a bounded domain, with dissipative feedback acting this time in the Neumann boundary conditions. The feedback system is now

$$\frac{\partial^2 w}{\partial t^2} = \Delta w \qquad \text{on } (0, \infty) \times \Omega \tag{4.1a}$$

$$\left.\begin{aligned} &w(0, x) = w_0(x) \in H^1(\Omega);\\ &w_t(0, x) = w_1(x) \in L_2(\Omega) \end{aligned}\right. \tag{4.1b}$$

$$\frac{\partial}{\partial \nu} w = -w_t \qquad \text{on } (0, \infty) \times \Gamma \tag{4.1c}$$

Let N denote the "Neumann map,"[3] i.e., the continuous map $L_2(\Gamma) \to H^{3/2}(\Omega)$ [N1, L–M1] which solves the elliptic problem corresponding to (4.1) (modulo an additive constant). It is defined by $f = Nv$, where $\Delta f = 0$ in Ω and $\partial f/\partial \nu = v$ on Γ, with $\int_\Gamma v \, d\Gamma = 0$, where f is the unique solution orthogonal in $L^2(\Omega)$ to the null space $\mathcal{N}(A)$ of constant functions. Moreover, one can see [L–T5, p. 64] that problem (4.1) corresponds to the feedback operator

$$\mathscr{A}_N = \begin{vmatrix} 0 & I \\ -A & -ANN^*A \end{vmatrix} \tag{4.2}$$

dissipative on $W \equiv \mathscr{D}(A^{1/2}) \times L_0^2(\Omega)$ and generator of a s.c. semigroup of contractions $e^{\mathscr{A}_N t}$ on W (counterpart of (1.12)–(1.13)). Here, $-A = \Delta$ with zero Neumann B. C. on $L_0^2(\Omega) = L^2(\Omega)/\mathcal{N}(A) = \{f \in L^2(\Omega) : \int_\Omega f \, d\Omega = 0\}$, $\mathcal{N}(A)$ begin the null space of A. With the usual condition $\int_\Omega w \, d\Omega = 0$ imposed on (4.1), then we have as is well known:

$$\text{gradient norm} = \int_\Omega |\nabla w|^2 \, d\Omega = \|A^{1/2} w\|_\Omega^2, \text{ equivalent to } \|w\|_{H^1(\Omega)} \tag{4.1d}$$

which we shall need in the sequel. While we refer to the end of the introduction for relevant literature on the question under study, our purpose here is to point out that for problem (4.1) new proofs can be given of both the strong stability result [Q–R1;S2; L1, Corollary 1] and the uniform, exponential stability result (as in [L1], generalizing [C1–2]), by means of the same techniques employed in the Dirichlet B. C. case in Sections 2 and 3. First, the operator theoretic proof of Theorem 1.1 based on the operator (1.12) in the Dirichlet case works equally well for the operator (4.2) in the Neumann case, thereby yielding another proof of strong stabilization of problem (4.1). Second, we shall provide in the present section a sketchy treatment o the uniform exponential energy decay result for problem (4.1), as a parallel counterpart of the analysis of Section 3. This way, we shall reprove Lagnese's main result [L1], under slightly weaker assumptions. (We use (1.21c) instead of (1.21b)).

THEOREM 4.1. *Let the bounded domain $\Omega \subset R^n$ possess a vector field $h = [h_1(x), ..., h_n(x)] \in C^2(\bar{\Omega})$ satisfying the following two assumptions:*

(H1′) *$h \cdot \nu \geqslant \text{const} = \gamma > 0$ on Γ [replacement of* (H1) *in the Dirichlet case]*

(H2′) *same as assumption* (H2) *in the Dirichlet case.*

Then there exist constants $C, \delta > 0$ such that, for any $[w_0, w_1] \in W =$

[3] We can likewise define N_1 by $N_1 v = f$, where $\Delta f = f$ in Ω, $\partial f/\partial \nu = v$ on Γ, $v \in L^2(\Gamma)$. Then, the analysis below works on the space $H^1(\Omega) \times L_2(\Omega)$ with

$$\mathscr{A}_{N_1} = \begin{vmatrix} 0 & I \\ -A & -A_1 N_1 N_1^* A_1 \end{vmatrix}$$

$\mathscr{D}(A^{1/2}) \times L_0^2(\Omega)$, *the corresponding solution of the feedback system* (4.1) *satisfies*

$$\left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_W \equiv \left\| e^{\mathscr{A}_N t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_W \leqslant Ce^{-\delta t} \left\| \begin{vmatrix} w_0 \\ w_t \end{vmatrix} \right\|_W, \qquad t \geqslant 0.$$

Proof. Since our proof is, in fact, a (much simplified) version of the same technique employed in Section 3 in the Dirichlet case, only the major elements will be pointed out. First, the second-order abstract model, corresponding to problem (4.1) is now

$$w_{tt} = -A[w + NN^* A w_t] \tag{4.3}$$

counterpart of (1.16). As in (3.2), we define

$$Q(t) = Q_1(t) + Q_2(t) \tag{4.4a}$$

where now $W = \mathscr{D}(A^{1/2}) \otimes L_0^2(\Omega)$ and

$$E(t) \equiv \|A^{1/2} w(t)\|_\Omega^2 + \|w_t(t)\|_\Omega^2 = \left\| e^{\mathscr{A}_N t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_W = \left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_W \tag{4.5}$$

$$Q_1(t) \equiv qtE(t), \qquad \text{as in (3.3a)} \tag{4.4b}$$

$$Q_2(t) \equiv -\tfrac{1}{2}((r - \operatorname{div} h)\, w_t(t), w(t))_\Omega \tag{4.4c}$$

with q and r positive constants to be chosen below in (4.24). By contraction of $E(t)$ we again obtain (3.4). Differentiating $Q(t)$ in t, using (4.3) and recalliing that $N^* A z = z|_\Gamma$ [L–T5], we find that the counterpart of (3.7) or (3.7′) is now

$$\int_{t_0}^\infty e^{-2\beta t} \frac{d}{dt} Q(t)\, dt = G_1(w) + G_2(w) - 2q \int_{t_0}^\infty te^{-2\beta t} \|w_t(t)\|_\Gamma^2\, dt \tag{4.6}$$

where

$$\begin{aligned} G_1(w) &\equiv G_{1,q,r,t_0,\beta}(t) \\ &= q \int_{t_0}^\infty e^{-2\beta t} E(t)\, dt - \frac{r}{2} \int_{t_0}^\infty e^{-2\beta t} [\|w_t(t)\|_\Omega^2 - \|A^{1/2} w(t)\|_\Omega^2]\, dt \end{aligned} \tag{4.10a}$$

$$\begin{aligned} G_2(w) &\equiv G_{2,r,t_0,\beta}(t) \\ &\equiv \frac{1}{2} \int_{t_0}^\infty e^{-2\beta t} [(w_t(t), w_t(t) \operatorname{div} h)_\Omega - (Aw(t), w(t) \operatorname{div} h)_\Omega]\, dt \\ &\quad + \frac{1}{2} \int_{t_0}^\infty e^{-2\beta t} (r - \operatorname{div} h)\, w_t(t), w(t))_\Gamma\, dt \end{aligned} \tag{4.10b}$$

Thus, as with Dirichlet case of Section 3, our goal now is to establish the counterpart of inequality (3.8), i.e.,

$$\int_{t_0}^{\infty} e^{-2\beta t} \frac{d}{dt} Q(t)\, dt \leqslant G_1(w) + G_2(w) - 2qt_0 \int_{t_0}^{\infty} e^{-2\beta t} \|w_t(t)\|_\Gamma^2\, dt$$

$$\leqslant -C^2_{q,r,t_0} \int_{t_0}^{\infty} e^{-2\beta t} E(t)\, dt + C'_{q,r,t_0} E(t_0) \tag{4.11}$$

for sufficiently large t_0. Using trace theory on (4.10b)

$$G_2(w) = \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} [(w_t, w_t \operatorname{div} h)_\Omega - (Aw, w \operatorname{div} h)]\, dt$$

$$+ \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|w_t(t)\|_\Gamma^2\, dt\right) + \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \|A^{1/2} w(t)\|_\Omega^2\, dt\right) \tag{4.12}$$

counterpart of (3.17), where—as in section 3—$\mathcal{O}$ denotes upper bound with a multiplicative constant *independent* of β and t_0. In order to achieve (4.11), the next step is to obtain—in Proposition 4.2 below—a *lower bound* on $\int_{t_0}^{\infty} e^{-2\beta t} \|w_t(t)\|_\Gamma^2\, dt$ which is the counterpart of the lower bound for $\int_{t_0}^{\infty} e^{-2\beta t} \|(\partial/\partial \nu) A^{-1} w_t\|_\Gamma\, dt \equiv \int_{t_0}^{\infty} e^{-2\beta t} \|(\partial/\partial \nu) p(t)\|_\Gamma\, dt$ obtained in (3.59) in the Dirichlet case. This will be accomplished, as in Section 3, by applying to the present Neumann case an adaptation of the same multiplier technique used in [LLT1] in the Dirichlet case. More precisely, the starting point now is the original equation

$$w_{tt} = \Delta w; \tag{4.13a}$$

$$w(0) = w_0; \qquad w_t(0) = w_1 \tag{4.13b}$$

$$\frac{\partial w}{\partial \nu} = -w_t \tag{4.13c}$$

instead of Eq. (3.23) for p in the Dirichlet case. Thus, we multiply (4.13) by $(h \cdot \nabla w) e^{-2\beta t}$, and integrate from t_0 to ∞[4]. Arguing as in Proposition 3.1 we obtain now the counterpart of Proposition 3.1.

[4] Compare with (3.23), where the unbounded term F is present. Indeed, the absence of F in the Neumann case greatly simplifies the analysis.

Proposition 4.1. *Let Ω possess a vector field $h(x) \in C^2(\bar{\Omega})$. Then for any $[w(t_0), w_t(t_0)] \in \mathscr{D}(\mathscr{A}_N)$ and any $\beta > 0$, we have*

$$\int_{t_0}^{\infty} e^{-2\beta t} \left[\int_{\Gamma} \left(\frac{\partial w}{\partial \nu} (h \cdot \nabla w) - \frac{1}{2} |\nabla w|^2 h \cdot \nu \right) d\Gamma \right] dt$$

$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} |w_t|^2 h \cdot \nu \, d\Gamma \, dt$$

$$= \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} H \nabla w \cdot \nabla w \, d\Omega \, dt$$

$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} [(w_t^2, \operatorname{div} h)_\Omega - (|\nabla w|^2, \operatorname{div} h)_\Omega] \, dt$$

$$+ e^{-2\beta t_0} (w_t(t_0), \nabla w(t_0) \cdot h)_\Omega - 2\beta \int_{t_0}^{\infty} e^{-2\beta t} (w_t, \nabla w \cdot h)_\Omega \, dt \tag{4.14}$$

with H as in (3.30).

We next use (4.13c) in the first integral on the left of (4.14) to obtain

$$\int_{\Gamma} \frac{\partial w}{\partial \nu} (h \cdot \nabla w) \, d\Gamma \leqslant \frac{M_h}{\varepsilon} \int_{\Gamma} w_t^2 \, d\Gamma + \varepsilon M_h \int_{\Gamma} |\nabla w|^2 \, d\Gamma, \; 2M_h = \max_{\Omega} |h| \tag{4.15}$$

Moreover, we use assumption (H1′) to get

$$\int_{\Gamma} |\nabla w|^2 h \cdot \nu \, d\Gamma \geqslant \gamma \int_{\Gamma} |\nabla w|^2 \, d\Gamma \tag{4.16}$$

and assumption (H2′) on the integral containing H. We thus obtain via (4.14)–(4.16)

$$\frac{M_h}{\varepsilon} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} w_t^2 \, d\Gamma \, dt + \varepsilon M_h \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} |\nabla w|^2 \, d\Gamma \, dt$$

$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} w_t^2 h \cdot \nu \, d\Gamma \, dt$$

$$\geqslant \frac{\gamma}{2} \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Gamma} |\nabla w|^2 \, d\Gamma \, dt + \rho \int_{t_0}^{\infty} e^{-2\beta t} \int_{\Omega} |\nabla w|^2 \, d\Omega \, dt$$

$$+ \frac{1}{2} \int_{t_0}^{\infty} e^{-2\beta t} (w_t^2 - |\nabla w|^2, \operatorname{div} h)_\Omega \, dt - 2M_h E(0) \tag{4.17}$$

since by contraction of $E(\ )$:

$$e^{-2\beta t_0} (w_t(t_0), \nabla w(t_0) \cdot h)_\Omega - 2\beta \int_{t_0}^{\infty} e^{-2\beta t} (w_t, \nabla w \cdot h)_\Omega \, dt.$$

Choosing now ε sufficiently small so that $((\gamma/2)-\varepsilon M_h)>0$, we can drop the term $\int_\Gamma |\nabla w|^2\, d\Gamma$ and (4.17) yields

PROPOSITION 4.2. *With a vector field h satisfying* (H1′) *and* (H2′) *we have for any $\beta>0$ and $[w(t_0), w_t(t_0)]\in \mathscr{D}(\mathscr{A}_N)$*

$$\frac{1}{2}\int_{t_0}^{\infty} e^{-2\beta t}[(w_t^2, \operatorname{div} h)_\Omega - (|\nabla w|^2, \operatorname{div} h)_\Omega]\, dt + \rho\int_{t_0}^{\infty} e^{-2\beta t}\int_\Omega |\nabla w|^2\, d\Omega\, dt$$
$$= \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t}\int_\Gamma w_t^2\, d\Gamma\, dt\right) + \mathcal{O}(E(t_0)). \tag{4.18}$$

We next return to G_2 in (4.12). By Green's theorem, (4.13c) and the identity $\nabla w\cdot\nabla(w \operatorname{div} h) = w\nabla(\operatorname{div} h)\cdot\nabla w + \operatorname{div} h\, |\nabla w|^2$, we compute

$$(w_t, w_t \operatorname{div} h)_\Omega - (Aw, w \operatorname{div} h)_\Omega$$
$$= (w_t^2 - |\nabla w|^2, \operatorname{div} h)_\Omega - (w_t, w \operatorname{div} h)_\Omega - (w, \nabla(\operatorname{div} h)\cdot\nabla w)_\Omega$$
$$= (w_t^2 - |\nabla w|^2, \operatorname{div} h) + \varepsilon_1 \mathcal{O}\left(\int_\Omega w_t^2 + |\nabla w|^2\, d\Omega\right) + \frac{1}{\varepsilon_1}\mathcal{O}\left(\int_\Omega w^2\, d\Omega\right)$$

$$(w_t, w_t \operatorname{div} h)_\Omega - (Aw, w \operatorname{div} h)_\Omega$$
$$= (w_t^2 - |\nabla w|^2, \operatorname{div} h)_\Omega + \varepsilon_1 \mathcal{O}(E(t)) + \frac{1}{\varepsilon_1}\mathcal{O}(\|w\|_\Omega^2). \tag{4.19}$$

By virtue of (4.19) and (4.18), the expression (4.12) for G_2 becomes

PROPOSITION 4.3. *For h as in Proposition* 4.2 *we have for any $\beta>0$*

$$G_2(w) = -\rho\int_{t_0}^{\infty} e^{-2\beta t}\int_\Omega |\nabla w|^2\, d\Omega\, dt + \varepsilon_1 \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t}E(t)\, dt\right)$$
$$+ \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t}\int_\Gamma w_t^2\, d\Gamma\, dt\right) + \mathcal{O}(E(t_0))$$
$$+ \frac{1}{\varepsilon_1}\mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t}\int_\Omega w^2\, d\Omega\, dt\right) \tag{4.20}$$

which is the counterpart of (3.65) *in Proposition* 3.6 *in the Dirichlet case.*

We now recall the equality (4.1d) for gradient norm and $A^{1/2}$-norm; let

$$\|A^{1/2}w\|_\Omega^2 = \int_\Omega |\nabla w|^2\, d\Omega. \tag{4.21}$$

Using (4.10a) for G_1 and (4.20) for G_2, we obtain

$$\begin{aligned}
&G_1(w)+G_2(w)-2qt_0\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Gamma} w_t^2\, d\Gamma\, dt \\
&= q\int_{t_0}^{\infty} e^{-2\beta t}E(t)\, dt+\varepsilon_1\mathcal{O}_1\left(\int_{t_0}^{\infty} e^{-2\beta t}E(t)\, dt\right) \\
&+\left(\frac{r}{2}-\rho\right)\int_{t_0}^{\infty} e^{-2\beta t}\, \|A^{1/2}w\|_{\Omega}^2\, dt \\
&-\frac{r}{2}\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Omega} w_t^2\, d\Omega\, dt+\mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Gamma} w_t^2\, d\Gamma\, dt\right) \\
&-2qt_0\left(\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Gamma} w_t^2\, d\Gamma\, dt\right)+\mathcal{O}(E(t_0)) \\
&+\frac{1}{\varepsilon_1}\mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Omega} w^2\, d\Omega\, dt\right)
\end{aligned}\tag{4.22}$$

counterpart of (3.68) in the Dirichlet case. We rewrite (4.22) as

$$\begin{aligned}
&G_1(w)+G_2(w)-2qt_0\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Gamma} w_t^2\, d\Gamma\, dt \\
&=\left[q+\varepsilon_1\mathcal{O}_1-\frac{r}{2}\right]\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Omega} w_t^2\, d\Omega\, dt \\
&+\left[q+\varepsilon_1\mathcal{O}_1+\frac{r}{2}-\rho\right]\int_{t_0}^{\infty} e^{-2\beta t}\, \|A^{1/2}w\|_{\Omega}^2\, dt \\
&+(\mathcal{O}_2+2qt_0)\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Gamma} w_t^2\, d\Gamma\, dt+\mathcal{O}(E(t_0)) \\
&+\frac{1}{\varepsilon_1}\mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t}\int_{\Omega} w^2\, d\Omega\, dt\right).
\end{aligned}\tag{4.23}$$

Thus, to achieve our goal of establishing inequality (4.11), we proceed as follows. With ρ given by Assumption (H.2′) (i.e. (1.21c)), we make $\varepsilon_1>0$ sufficiently small and choose positive constants q and r such that

$$\begin{aligned}
&q+\varepsilon_1\mathcal{O}_1-\frac{r}{2}<0 \\
&q+\varepsilon_1\mathcal{O}_1+\frac{r}{2}-\rho<0
\end{aligned}\tag{4.24}$$

Next, we choose t_0 large enough that

$$\mathcal{O}_2 - 2qt_0 < 0. \tag{4.25}$$

This way, we obtain from (4.23) via (4.24)–(4.25)

$$\begin{aligned} &G_1(w) + G_2(w) - 2qt_0 \int_{t_0}^{\infty} e^{-2\beta t} \int_\Gamma w_t^2 \, d\Gamma \, dt \\ &\quad \leqslant -C_{q,r,t_0}^2 \int_{t_0}^{\infty} e^{-2\beta t} E(t) \, dt + C'^2_{q,r,t_0} E(t_0) + \frac{1}{\varepsilon_1} \mathcal{O}\left(\int_{t_0}^{\infty} e^{-2\beta t} \int_\Omega w^2 \, d\Omega \, dt \right). \end{aligned} \tag{4.26}$$

In order to establish (4.11), we need a counterpart of Proposition 3.7, i.e.,

PROPOSITION 4.4N. *For any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for any $\beta > 0$ and $t_0 > 0$ we have*

$$\int_{t_0}^{\infty} e^{-2\beta t} \, \|w(t)\|_\Omega^2 \, dt \leqslant \varepsilon \int_{t_0}^{\infty} e^{-2\beta t} \, \|w_t(t)\|_\Omega^2 \, dt + (t_0 + 1) \, C_\varepsilon E(t_0).$$

The proof of Proposition 4.4N is given in [L1] or alternatively can be obtained by using the same operator approach as in Proposition 3.7 of the present paper. Then (4.26), together with Proposition 4.4, completes the proof of (4.11), hence of Theorem 4.1.

APPENDIX A: PROOF OF LEMMA 3.3

Dropping throughout the explicit dependence on t, we write $w = AA^{-1}w = -\Delta(A^{-1}w)$ and apply Green's second theorem

$$\begin{aligned} (w, h \cdot \nabla(DD^* w_t))_\Omega &= -(\Delta(A^{-1}w), h \cdot \nabla(DD^* w_t))_\Omega \\ &= -(A^{-1}w, \Delta(h \cdot \nabla(DD^* w_t))_\Omega \\ &\quad - \left(\frac{\partial}{\partial \nu}(A^{-1}w), h \cdot \nabla(DD^* w_t) \right)_\Gamma \\ &\quad + \left(A^{-1}w, \cancel{\frac{\partial}{\partial \nu}} (h \cdot \nabla(DD^* w_t)) \right)_\Gamma \end{aligned} \tag{A.0}$$

where cancellation of the last term occurs since $A^{-1}w \in \mathscr{D}(A)$, and so vanishes on Γ. We first consider the second term in the right-hand side of (A.0): to this end, we use the assumption on the vector field h that $h|_\Gamma$ is parallel to v at each point of Γ, and so $h|_\Gamma = k(\sigma)v$, $\sigma \in \Gamma$, for some scalar $k(\sigma) \in C(\Gamma)$. Thus, by (1.10),

$$-\left(\frac{\partial}{\partial v}(A^{-1}w), h \cdot \nabla(DD^*w_t)\right)_\Gamma = (D^*w, h \cdot \nabla(DD^*w_t))_\Gamma$$

$$= \left(D^*w, k\frac{\partial}{\partial v}(DD^*w_t)\right)_\Gamma \qquad \text{a.e. in } t. \tag{A.1}$$

But by Green's second theorem with $D(kD^*w) = kD^*w$ on Γ (by definition of D in (1.5)), we compute

$$\left(D^*w, k\frac{\partial}{\partial v}(DD^*w_t)\right)_\Gamma = \left(D(kD^*w), \frac{\partial}{\partial v}(DD^*w_t)\right)_\Gamma$$

$$= (D(kD^*w), \Delta(DD^*w_t))_\Omega - (\Delta(D(kD^*w), DD^*w_t)_\Omega$$

$$+ \left(\frac{\partial}{\partial v}(D(kD^*w)), D^*w_t\right)_\Gamma$$

$$= \left(\frac{\partial}{\partial v}(DD^*w), kD^*w_t\right)_\Gamma \qquad \text{a.e. in } t. \tag{A.2}$$

Moreover

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial v}(DD^*w), kD^*w\right)_\Gamma = \left(\frac{\partial}{\partial v}(DD^*w_t), kD^*w\right) + \left(\frac{\partial}{\partial v}(DD^*w), kD^*w_t\right)_\Gamma$$

$$= 2\left(D^*w, k\frac{\partial}{\partial v}(DD^*w_t)\right)_\Gamma \qquad \text{a.e. in } t \tag{A.3}$$

by (A.2). Inserting (A.3) into (A.1), we finally obtain

$$-\left(\frac{\partial}{\partial v}(A^{-1}w), h \cdot \nabla(DD^*w_t)\right)_\Gamma = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial v}(DD^*w), kD^*w\right)_\Gamma \qquad \text{a.e. in } t \tag{A.4}$$

as desired.

We next consider the first term on the right-hand side of (A.0). With g a scalar, say $H^2(\Omega)$-function, one preliminarily obtains by direct computations

$$\Delta(h \cdot \nabla g) = \sum_{i=1}^{n} g_{x_i} \Delta h_i + h \cdot \nabla(\Delta g) + 2\Delta g \operatorname{div} h + Sg \tag{A.5}$$

where S is a second-order operator on g (e.g., for 2 variables $Sg = 2[h_{2x_1} g_{x_2x_1} + h_{1x_2} g_{x_1x_2} - h_{1x_1} g_{x_2x_2} - h_{2x_2} g_{x_1x_1}]$). Specializing (A.5) with $g = DD^*w_t$, whereby $\Delta g = \Delta(DD^*w_t) = 0$ in Ω, we obtain

$$\Delta(h \cdot \nabla(DD^*w_t)) = \sum_{i=1}^{n} g_{x_i} \Delta h_i + Sg. \tag{A.6}$$

Thus the first term on the right of (A.1) can be written as

$$(A^{-1}w, \Delta(h \cdot \nabla(DD^*w_t))_\Omega = \sum_{i=1}^{n} (A^{-1}w, g_{x_i} \Delta h_i)_\Omega + (A^{-1}w, Sg)_\Omega. \tag{A.7}$$

Using the right-hand side of (A.7), we shall show below that (A.7) implies

$$|(A^{-1}w, \Delta(h \cdot \nabla(DD^*w_t))_\Omega| \leqslant C \|w\|_\Omega \|D^*w_t\|_\Gamma \qquad \text{a.e. in } t. \tag{A.8}$$

Thus, using identity (A.3) and estimate (A.8) on the right-hand side of (A.1), we obtain (3.48) as desired. Hence, the proof of Lemma 3.3 is complete, once we establish inequality (A.8) as a consequence of (A.7).

Proof of (A.8). It is based on the following generalization of Green's second theorem, to be justified at the end. Let S_1 be the principal part of a second-order differential operator on Ω with smooth coefficients $S_1 = \sum_{i,j=1}^{n} a_{ij}(\partial/\partial x_i)(\partial/\partial x_j)$. Then, for u, v sufficiently smooth, we have

$$(u, S_1 v)_\Omega - (S_2 u, v)_\Omega = (u, (F_1 v)_\Gamma)_\Gamma - ((F_2 u)_\Gamma + (F_3 u, v)_\Omega - (u, F_4 v)_\Omega \tag{A.9}$$

where S_2 is a suitable second-order operator and F_i, $i = 1,..., 4$ are suitable first-order operators on $L_2(\Omega)$. With $S_1 = S$ given in (A.6), $u = A^{-1}w$ and $v = g = DD^*w_t$ [so that $v|_\Gamma = D^*w_t$], then (A.9) specializes to

$$\begin{aligned}(A^{-1}w, Sg)_\Omega &\\ = (S_2 A^{-1}w, DD^*w_t)_\Omega &+ (A^{-1} w, (F_1 g)_\Gamma)_\Gamma - ((F_2 A^{-1}w)_\Gamma, D^*w_t)_\Gamma \\ &+ (F_3 A^{-1}w, DD^*w_t)_\Omega - (A^{-1}w, F_4 DD^*w_t)_\Omega \end{aligned} \tag{A.10}$$

with cancellation since $A^{-1}w \in \mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Thus, for $w \in L_2(\Omega)$, we have: $S_2 A^{-1}w \in L_2(\Omega)$; $F_i A^{-1}w \in H^1(\Omega)$ (by [LM1, I, Proposition 12.1 p. 85]; $(F_2 A^{-1}w)_\Gamma \in H^{1/2}(\Gamma)$; $DD^*w_t \in H^{1/2}(\Omega)$ (by Theorem 1.1(iii) and (1.7a) with $s = 0$), and finally $g_{x_i} \Delta h_i$ and $F_4 DD^*w_t$ both in $[H_{00}^{1/2}(\Omega)]' \cap H^{-1/2-\varepsilon}(\Omega)$ continuously (again by [LM1, I, Proposition 12.1 p. 85]). Thus, by (A.10)

$$(A^{-1}w, g_{x_i} \Delta h_i)_\Omega + (A^{-1}w, Sg)_\Omega \leqslant C \|w\|_\Omega \|D^*w_t\|_\Gamma \qquad \text{a.e. in } t \tag{A.11}$$

and (A.8) then follows from the right-hand side of (A.7) via (A.11). The justification of identity (A.9) is as follows. If F is the principal part of a first-order differential operator with smooth coefficients: $F=\sum_{i=1}^{n} a_i(\partial/\partial x_i)$, then the divergence theorem applied to the vector field $V=[a_1 uv, a_2 uv,..., a_n uv]$, u, v smooth functions, with $\operatorname{div} V=(\sum (\partial/\partial x_i)a_i)uv+(Fu)v+u(Fv)$ yields

$$(Fu, v)_\Omega+(u, Fv)_\Omega+\left(\sum \frac{\partial a_i}{\partial x_i} u, v\right)_\Omega=\int_\Gamma uv[a_1,..., a_n]\cdot \nu\, d\Gamma. \tag{A.12}$$

Next, if $\mathscr{F}_1$ and $\mathscr{F}_2$ are two first-order differential operators as F before, then (A.12) applied for $F=\mathscr{F}_1$ gives

$$\begin{aligned}(\mathscr{F}_1\mathscr{F}_2 u, v)_\Omega&+(\mathscr{F}_2 u, \mathscr{F}_1 v)_\Omega+\left(\left(\sum \frac{\partial a_i^1}{\partial x_i}\right)\mathscr{F}_2 u, v\right)_\Omega\\ &=\int_\Gamma (\mathscr{F}_2 u)\, v[a_1^1,..., a_n^1]\cdot \nu\, d\Gamma.\end{aligned} \tag{A.13}$$

But (A.12) applied now for $F=\mathscr{F}_2$ yields

$$\begin{aligned}(\mathscr{F}_2 u, \mathscr{F}_1 v)_\Omega&+(u, \mathscr{F}_2\mathscr{F}_1 v)_\Omega+\left(\left(\sum \frac{\partial a_i^2}{\partial x_i}\right) u, \mathscr{F}_1 v\right)_\Omega\\ &=\int_\Gamma u(\mathscr{F}_1 v)[a_1^2,..., a_n^2]\cdot \nu\, d\Gamma.\end{aligned} \tag{A.14}$$

Inserting (A.14) into (A.13) gives

$$\begin{aligned}(\mathscr{F}_1\mathscr{F}_2 u, v)_\Omega&-(u, \mathscr{F}_2\mathscr{F}_1 v)_\Omega\\ &=\int_\Gamma (\mathscr{F}_2 u)\, v[a_1^1,..., a_n^1]\cdot \nu\, d\Gamma-\int_\Gamma u(\mathscr{F}_1 v)[a_1^2,..., a_n^2]\cdot \nu\, d\Gamma\\ &\quad+\left(\left(\sum \frac{\partial a_i^2}{\partial x_i}\right) u, \mathscr{F}_1 v\right)_\Omega-\left(\left(\sum \frac{\partial a_i^1}{\partial x_i}\mathscr{F}_2 u, v\right)\right)_\Omega.\end{aligned} \tag{A.15}$$

Finally, since the second-order differential operator S_1 introduced above (A.9) can be written as $S_1=\sum_{i=1}^{n}\mathscr{F}_{1i}\mathscr{F}_{2i}$ for suitable first-order operators $\mathscr{F}_{1i}$ and $\mathscr{F}_{2i}$ (principal part), then (A.9) follows from (A.15). The proof of Lemma 3.3 is complete. ∎

APPENDIX B: STRICTLY CONVEX DOMAINS SATISFY THE VECTOR FIELD ASSUMPTION

The content of the present Appendix is entirely due to Professor Ennio De Giorgi, Scuola Normal Superiore, Pisa, and was communicated to us by him orally during the summer 1985. It is inserted here with his consent.

(1) Let $f(x)$ be a strictly convex function $R^n \to R$ of class C^3. Then, the set $D \subset R^n$

$$D \equiv \{x \in R^n; k_1 \leqslant f(x) \leqslant k_2\}, k_i \text{ real}; k_1 < k_2$$

is bounded with C^2-boundary ∂D. Moreover, D is (a) either strictly convex if $k_1 < \min_{x \in R^n} f(x)$, or else (b) is "donot-like," i.e., $D = D_2 \backslash D_1$, the set difference of two strictly convex sets D_2, D_1 of R^n, with $\bar{D}_1 \nsubseteq D_2$. In case (a), we have $f(\partial D) \equiv k_2$, while in case (b), we have $\partial D = \partial D_1 \cup \partial D_2$, ∂D_1 and ∂D_2 disjoint, and $f(\partial D_1) \equiv k_1$ while $f(\partial D_2) \equiv k_2$. In any case, for $x \in \partial D$ we have $\nabla f(x) \perp \partial D$. Thus, the vector field h defined by

$$h(x) \equiv \nabla f(x) \qquad x \in D = \bar{D}$$

satisfies conditions close to the *vector field assumption* on $D = \bar{D}$: (i) Jacobian matrix $(h) =$ Hessian matrix $(f) \geqslant 0$ on $\bar{D} = D$ (ii) $h \perp \partial D$ on the boundary.

(2) Conversely, let D be a strictly convex bounded closed set of R^n, with C^2-boundary ∂D. The following procedure constructs a strictly convex, C^∞-function $F(x)$, such that

$$D = \{x \in R^n: 0 < k \leqslant F(x) \leqslant 1\}$$
$$F(\partial D) \equiv 1$$

so that $h(x) \equiv \nabla F(x)$ satisfies the vector field assumption on D.

Let O (origin) be an internal point of D and define a function $w(x): D \to R$ by setting $w(O) = 0$ $w(B) = 1$ for B an arbitrary boundary point, $B \in \partial D$, while w is defined linearly (by homogeneity on the segment OB. Thus, w is defined on all of D, but it is not strictly convex and is not c^1 at the origin. Its square $w^2(x)$ is strictly convex on D but still not c^2 at the origin and is therefore compared with the paraboloid $\varepsilon |x|^2 + t$, $0 < t < 1$, with $0 < \varepsilon$ sufficiently small. Setting $a \vee b \equiv \max[a, b]$, we have

$$w^2(x) \vee [\varepsilon |x|^2 + t] = \text{strictly convex on } D \text{ for each } t, \text{ smooth except at the intersection points } w^2(x) = \varepsilon |x|^2 + t.$$

Now if $\phi(t)$ is a $C^\infty(R)$-function, $\phi > 0$, with compact support, $\operatorname{supp} \phi \in [0, 1]$, normalized by $\int_{-\infty}^{\infty} \phi(t)\, dt = 1$, then the function

$$F(x) \equiv \int_{-\infty}^{\infty} \{w^2(x) \vee [\varepsilon\, |x|^2 + t]\}\, \phi(t)\, dt$$

$$\equiv w^2(x) \int_{-\infty}^{w^2(x) - \varepsilon |x|^2} \phi(t)\, dt + \int_{w^2(x) - \varepsilon |x|^2}^{+\infty} (\varepsilon\, |x|^2 + t)\, \phi(t)\, dt$$

is smooth and strictly convex on D. Moreover, if ε is chosen sufficiently small so that $\varepsilon\, |x|^2 + t \leqslant 1$ for $t \in \operatorname{supp} \phi$, then for any boundary point $B \in \partial D$ of D we have:

$$F(B) \equiv \int_{-\infty}^{\infty} 1\phi(t)\, dt \equiv 1, \qquad \text{while } 0 < F(x) \leqslant 1, \quad x \in \bar{D}$$

and F has the required properties, since ∂ let $H(x) > 0$ on $\bar{D}$ (computations omitted).

(3) Similarly, given $D = D_2 \backslash D_1$, with D_1, D_2 strictly convex, $\bar{D}_1 \nsubseteq D_2$, and with boundary $\partial D = \partial D_2 \cup \partial D_1$ of class C^2, ∂D_2 and ∂D_1 disjoint, one can construct a smooth (C^2)-function $F(x)$ such that

$$D = \{x \in R^n : k_1 \leqslant F(x) \leqslant k_2\}$$

$$F(\partial D_1) \equiv k_1,\ F(\partial D_2) \equiv k_2$$

so that $h(x) \equiv \nabla F(x)$ satisfies the vector field condition on D.

Note added in proof. In the analysis above—based on the multiplier $e^{-\beta t} h \cdot \nabla p$—which leads from the hyperbolic problem (3.23) for p to the *lower bound* for $\int_\Sigma (\partial p/\partial \nu)^2\, d\Sigma$ in (3.59) (recall $\partial p/\partial \nu = -D^* w_t$ (3.19)), it is the presence of the non-homogeneous term F in (3.23) that forced the assumption that h is parallel to ν on Γ; see that Lemma 3.3 is needed to prove Proposition 3.2. Thus, if one considers the homogeneous problem "without F" instead, i.e., the problem

$$\text{(H.P.)} \qquad \begin{aligned} & y_{tt} = \Delta y && \text{in } (0, T) \times \Omega = Q_T \\ & y|_{t=0} = y_0, \quad y_t|_{t=0} = y_1 && \text{in } \Omega \\ & y \equiv 0 && \text{in } (0, T) \times \Gamma = \Sigma_T \end{aligned}$$

with $0 < T < \infty$, and applies the same argument as above—obviously with multiplier $h \cdot \nabla y$ this time, i.e., $\beta = 0$, since T is finite—then the assumption that h be parallel to ν on Γ is *not* needed at all. This way, for *any* $h \in C^1(\bar{\Omega})$, one obtains the following specialized version of identity (3.54)

$$\int_{\Sigma_T} \frac{\partial y}{\partial \nu} (h \cdot \nabla y)\, d\Sigma - \frac{1}{2} \int_{\Sigma_T} \left(\frac{\partial y}{\partial \nu}\right)^2 h \cdot \nu d\Sigma = \int_{Q_T} H \nabla y \cdot \nabla y dQ + \frac{1}{2} \int_{Q_T} (y_t^2 - |\nabla y|^2) \operatorname{div} h dQ$$
$$+ [(y_t, h \cdot \nabla y)_\Omega]_{t=0}^{t=T}; \tag{3.54A}$$

hence the following specialized version of the lower bound for the normal derivative in (3.59) (recall $|\partial y/\partial \nu| = |\nabla y|$) under (1.21c):

$$M_h \int_{\Sigma_T} \left(\frac{\partial y}{\partial \nu}\right)^2 d\Sigma \geqslant \rho \int_{Q_T} |\nabla y|^2 \, dQ + \frac{1}{2} \int_{Q_T} (y_t^2 - |\nabla y|^2) \operatorname{div} h dQ - 2M_h E(0), \quad (3.59\text{A})$$

where $2M_h = \max_\Omega |h|$ and $E(t) \equiv \int_\Omega y_t^2 + |\nabla y|^2 \, d\Omega \equiv E(0)$ for the conservative problem (H.P.).

We choose now the simplest h, i.e., $h(x) = x - x_0$, $x_0 \in R^n$ (so that $H(x) \equiv \text{Identity}$, $\rho = 1$, $\operatorname{div} h \equiv n = \dim \Omega$) and use

$$\left| \int_0^T \int_\Omega (y_t^2 - |\nabla y|^2) \, dQ \right| = |[(y_t, y)_\Omega]_0^T| \leqslant CE(0) \qquad (*)$$

(the identity without the absolute value is obtained by multiplying the equation of (H.P.) by y and integrating by parts; the inequality on the right follows by use of Poincaré inequality). Then, (3.59A) reduces to

$$M_h \int_{\Sigma_T} \left(\frac{\partial y}{\partial \nu}\right)^2 d\Sigma \geqslant \int_0^T \int_\Omega |\nabla y|^2 \, d\Omega dt - C_{1h} E(0),$$

from which we likewise obtain via (*):

$$M_h \int_{\Sigma_T} \left(\frac{\partial y}{\partial \nu}\right)^2 d\Sigma \geqslant \int_0^T \int_\Omega y_t^2 \, d\Omega dt - (C_{1h} + C) \, E(0).$$

Summing up the last two equations gives

$$2M_h \int_{\Sigma_T} \left(\frac{\partial y}{\partial \nu}\right)^2 d\Sigma \geqslant TE(0) - (2C_{1h} + C) \, E(0).$$

Thus, there are positive constants c, T_0 (easily computable in the analysis above) such that for all $T > T_0$:

$$\int_{\Sigma_T} \left(\frac{\partial y}{\partial \nu}\right)^2 d\Sigma \geqslant c(T - T_0) \, E(0). \qquad (\#)$$

In this explicit form, the lower bound (#) is derived in F. Ho (1986, personal communication to the authors by J. L. Lions). The opposite inequality

$$\int_{\Sigma_T} \left(\frac{\partial y}{\partial \nu}\right)^2 \leqslant C_T E(0)$$

is instead valid for all $T > 0$, and was first proved by the authors in [L.T.2 Theorem 2.3 and Remark 2.1, p. 279], (by operator methods) and, independently, by Lions [L.4]. It is reproved in [LLT.1] by the same multiplier method used here. Inequality (#) is the key "Lemma" in the recent result by Lions (Controlabilité exacte de systèmes distribues, *C. R. Acad. Sci.*, 1986), obtained by a *direct* approach, not via stabilization: exact controllability on $L^2(\Omega) \times H^{-1}(\Omega)$ for problem (1.1) with $u \in L_2(\Sigma_T)$ acting on the whole boundary is always true, with *no* geometrical conditions on Ω. This improves the geometrical conditions aspect of our Theorem 1.3. An extension of inequality (#) to a proper portion of the boundary, and a consequent extension of the exact controllability result in $L^2(\Omega)\times H^{-1}(\Omega)$ for problem (1.1) when u in (1.1c) acts only on a portion of the boundary—in which case geometrical conditions on Ω are needed!—are given (by virtue also of a direct approach, though different from Lions) in R. Triggiani "Exact boundary controllability on $L_2(\Omega)\times H^{-1}(\Omega)$ of the wave equation with Dirichlet boundary control acting on a portion of the boundary $\partial\Omega$, and related problems," 1986 preprint; presented at IFIP Workshop on "Boundary Control and Boundary Variations," held at Université de Nice, France, June 10–13, 1986; and also at the "Conference on Distributed Parameter Systems," held in Vorau, Austria, July 6–12, 1986.

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